THE UNIVERSITY OF ALBERTA

OPTIMAL CONTROL OF DYNAMIC SYSTEMS

A THESIS

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by

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INTRODUCTION

STATEMENT OF RESEARCH PROBLEM

To date, control design is generally based on an analysis approach. That is to say, the system is first constructed and then analyzed to see if the performance is satisfactory. In cases where unsatisfactory performance is incurred, the system is modified in some manner and then re-analyzed. This procedure may be satisfactory in cases where there are few control parameters to manipulate. Cases occur, however, where several independent forcing functions may exist. It may also be the case, that a nonlinear controller could out perform a linear controller.

This research is concerned with the problem of formulating and computing optimal control laws for the control of more complex dynamic systems. The optimization of a dynamic system, subject to a given performance criterion, and a known mathematical description of the plant and its surroundings, is carried out.

There is available, a number of techniques for evaluating optimal control laws. This information is generally found in mathematical and engineering papers published in various journals. To date, concise, complete results are difficult to find for the continuous-time case. The purpose of this paper is to consolidate, extend and compare results of modern ideas, in the control field of a dynamic system under a unified approach. A complete analysis is carried out using classical methods of calculus of variations.

STATEMENT OF THE GENERALIZED CONTROL PROBLEM

A dynamic system is one whose future behaviour depends on its past, as well as on its present and future inputs. The influence of a system's past, can be specified by a set of parameters which are termed the state variables to determine the future behaviour of the system. The outputs can be expressed as a function of the state variables so that if x(s) represent the system states, u(s), the control inputs, y(s) the outputs and w(s) the disturbances, then the dynamic system can be described by the following set of differential equations

$$\frac{dx(s)}{ds} = f\,(s;x(s),u(s),w(s))$$

$$y(s) = h((s),x(s)w(s))$$

The control of a dynamic system is a matter of varying u(s) in such a manner that the behaviour of the system is in some sense best. This optimization is carried out to optimize a specified performance criterion.

Performance criterions in the past have been used to specify error constants in a steady state situation. We now generalize the performance criterion, to be given by the following integral function

$$V = \mathcal{V}(x(T),T) + \int_t^T L(s;u(s),y(s),w(s),T)\,ds$$

where t is the initial time of the system and T is the terminal time.. In cases where stochastic inputs and disturbances are considered the expected value of V(s;x(s)u(s)) must be specified. L(s;T,u(s),y(s),w(s)) is called the loss function or cost function

of the dynamic system. It is not necessary that T be fixed; it may rather be that time at which a condition of the form

$g(T, x(s)) = 0$ becomes satisfied.

The inputs to the controller can be considered a set of parameters u(t) which are statistically related to w(s), $t \leq s \leq T$ and make central decisions on that basis.

Since the system behaviour, as characterized by x(s), $t \leq s < T$, depends on x(t) as well as u(s) and w(s), $t \leq s < T$, the control law k specifying u(t) must be a function of x(t) as well as u(t), t and possibly T:

$$u(t) = k(t, x(t), \boldsymbol{v}(t), T)$$

The problem is to select the control law which maintains along the entire trajectory x(s), $t \leq s < T$ the maximum performance criterion.

The specialized control problem described ~~alone~~ above is represented in Figure 1. It represents the control of a nonlinear, time-varying multi-input, multi-output dynamic system subject to stochastic disturbance.

CONTROL AND OPTIMIZATION OF DYNAMIC SYSTEMS - THE VARIATIONAL APPROACH

In this report, the variational approach is used to determine optimal controllers for a dynamic system. In Section I, Part I basic necessary conditions are developed for the generalized control problem using the variational approach to optimize the performance criterion subject to the constraints of the plant. These basic necessary

conditions are just the Euler Lagrange equations of the classical variational problem in n dimensions. In addition, boundary conditions which the optimal system must obey are developed. A second necessary condition is developed following the classical ideas of Weierstrass and forming the well-known Weierstrass E - function. This in turn, by using the quadratic remainder of the Taylor series, gives Legendres well-known necessary condition. Using a Legendre transformation, the Euler Lagrange equations are reduced to a canonical form which in turn must obey the Hamilton Jacobi partial differential equation given in Section I, Part 2. Hence an additional necessary condition for optimum is that the variation of the Hamiltonian be equal to zero.

The concepts of controllability and observability are developed in Section I, Part 2. Controllability and observability are sufficiency conditions for a linear time varying system. These sufficiency conditions can be thought of in the same light as stability of a dynamic system.

The Euler Lagrange equations form a system in general of nonlinear differential equations. Because of this, and because of the fact that virtually no theory has been developed for the general solutions of nonlinear equations, we are unable to solve the control problem in a general sense. If nonlinear equations were encountered, digital techniques would have to be employed and each problem would have to be assessed and solved on its own merits. Although we cannot solve nonlinear systems in general, solutions

are available for linear time varying systems. These solutions are given in Section II, Part 1 and 2, where the solution of Part 1 gives a new method of solving stationary linear problems and Part 2 gives a method for solving non-stationary linear problems. The computational advantage, and the more general solution (i.e., transients considered) of these methods over the older methods of using transform techniques, provide an excellent method of solving complex problems. These methods are particularly suited to digital techniques, although not treated as such in this report.

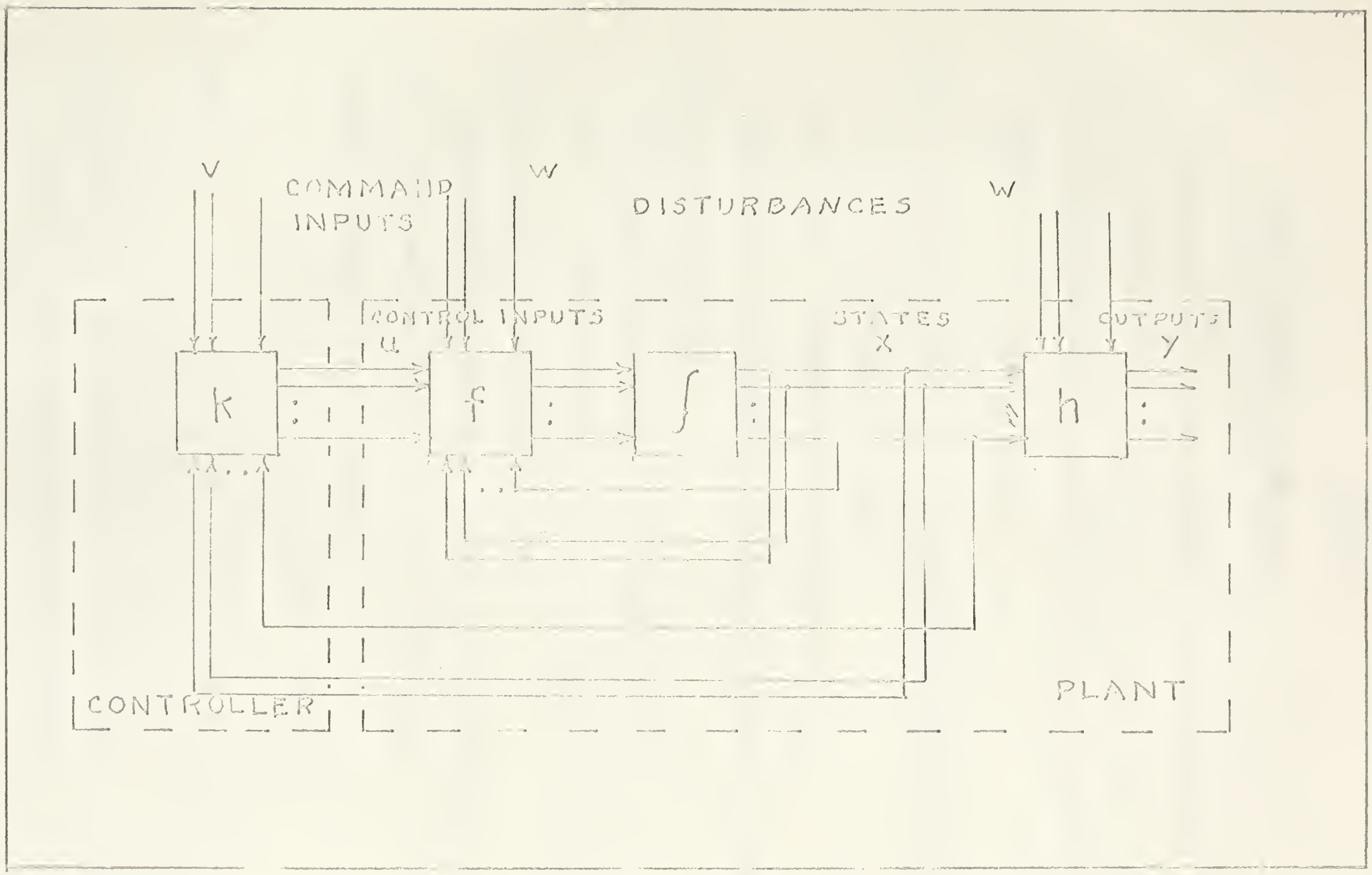
The solution of a linear system in no way guarantees that the control law is linear. Cases where linear systems are governed by nonlinear controllers are illustrated in Section II, Part 4. It is possible, however, with an additional assumption to the system (i.e., a quadratic loss function of the form given in Section II, Part 1 and 2) to form a controller which must be linear. Having formed the optimal control problem, as done in Section II, Part 1 and 2, there are several possible ways to solve this problem. If the initial time t is held fixed or specified for some value and successive values of T are investigated, the control law can be observed as it approaches its limit. This is known as the Floating time to go problem. If we now consider a fixed value of T, and successive values of t, another solution known as the Shrinking time to go problem, can be solved.

In Section II, Part 3, a somewhat more elegant method of solving the Shrinking time to go problem subject to natural boundary conditions, is suggested by using a method formulated by Kalman. This method assumes that a linear relationship between $\xi(t)$ and $x(t)$ exists and holds true at $t = T$. If this is the case, a differential equation can be formed for $P(t)$ where $\xi(t) = P(t)x(t)$. This differential equation assumes the well-known form of the Riccati equation and shall be denoted as such throughout this paper. Kalman's technique when applicable, has the advantage that we can treat the two-point boundary problem as an initial value problem, hence, well-known theory of ordinary differential equations can be applied. Also developed for this problem is a very useful stability theorem. This theorem states that if in addition to the conditions of the problem as stated in Section II, Part 3, the plant is also uniformly completely controllable and uniformly completely observable, then the optimal regulator is asymptotically stable. Thus, rather than check the stability of the regulator, it is only necessary to check whether or not the plant is uniformly, completely controllable and observable. In addition, Kalman's technique provides a simple way in which to calculate solutions for steady state problems. This is done by equating the right-hand side of the Riccati equation to zero and solving the resulting quadratic, algebraic equations for the nonnegative definite solution.

The solution of these algebraic equations is particularly suited to analog techniques. Further as shown in Section III, Part 1 and 2, the regulator problem is under duality relations equivalent to the linear filter and prediction problem.

In Section II, Part 4, methods of treating constraints for a dynamic system are considered. In addition necessary conditions for a linear plant subject to the constraint $|u_j| \leq 1$ are developed for minimum time, fuel and energy. The minimum fuel problem is then solved completely by use of a phase plane for a second order system.

In Section III, the linear filter problem is treated from a state transition method. The well-known Wiener-Hopf equations are reduced to a system of differential equations. In addition to being able to solve time varying systems, this method provides an easy computational way to solve the more well-known stationary problem. In Example Problem III,1-1, a stationary filter is solved.



GENERALIZED CONTROL PROBLEM

FIG. 1

SECTION I

Part I A BASIC NECESSARY CONDITION

The most general control problem is investigated and a basic necessary condition is developed for an optimal plant. The plant is described by the non-linear vector differential equation (1.1).

$$\frac{dx(s)}{ds} = f(s;x(s),u(s),w(s)) \quad \text{.................(1.1)}$$

where $f(s;x(s),u(s),w(s))$ is an n-dimensional vector function.

$x(s)$ is an n-dimensional vector.

$u(s)$ is an m-dimensional vector where $1 \leq m \leq n$

$w(s)$ is an r-dimensional vector which defines the disturbances that occur in the plant.

The output of the plant $y(s)$ is given by (1.2)

$$y(s) = h(s;x(s),w(s)) \quad \text{.......................(1.2)}$$

where $y(s)$ is an p-dimensional vector.

The general performance criterion "V" is specified for some future time from the present time "t" to that future time "T" (possibly infinity). is of the form.

$$V = \int_t^T L(s;T,x(s),u(s),w(s))ds + \nu(T,x(T)) \quad \text{.....(1.2)}$$

where "L" and "ν" are scalar functions of the vectors $x(s)$,$u(s)$,and $w(s)$.

It should be noted at this point that the vector function $h(s;x(s),w(s))$ is incorporated into the performance criterion.

In general we may specify a boundary condition at time "T".

$$g(T,x(T)) = 0 \quad \text{...............................(1.4)}$$

The current system state $x(t)$ is assumed known and the controller input $v(t)$ is considered to specify completely $w(t)$ for all $t \leq s \leq T$. For a given $x(t)$ and $v(t)$ it is required to find the function $u(t)$ for all $t \leq s \leq T$ which maximizes "V". The control law "k" specifying $u(s)$ must be a function of $x(s)$ as well as $v(s)$, s and possibly T. Therefore let

$$u(s) = k(s;x(s),v(s),T) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.5)$$

where "k" is a vector function and is the optimal control law sought.

Conditions on the system

(1) $u(s)$ is a function of class D^{o}.

(2) $f(s;x(s),u(s),w(s))$ is of class C' with respect to $x(s)$ and $u(s)$ is of class D^{o} with respect to "s" in the region R of interest in the $s,x(s),u(s)$ space.

(3) L is of class C' with respect to $x(s)$ and $u(s)$ and of class D^{o} with respect to "s" in R.

Note: L will later become of class C^2 with respect to $x(s)$ and $u(s)$ due to Legendre's necessary condition.

(4) The system is to be controllable with respect to the inputs $u(s)$ in R.

Note: The term controllable and conditions necessary for controllable plants for a linear system will be treated in detail later.

We now formulate a new performance criterion "V" where

$$V^{\#} = \int_t^T \left[L(s;T,x(s),u(s),w(s)) + \lambda(s) \left[\frac{dx(s)}{ds} - f(s;x(s),u(s),w(s)) \right] \right] ds + \mathcal{V}(T,x(T)) \quad \ldots\ldots\ldots\ldots\ldots\ldots (1.6)$$

where $\lambda(s)$ is the Lagrange multiplier.

Note: Finding a stationary point of "$V^{\#}$" with respect to the variations in $u(s), x(s)$, $\lambda(s)$, and $x(T)$ unconstrained is the same as finding a stationary point of "V" with respect to the variations $x(s)$, $u(s)$, $\lambda(s)$, and $x(T)$ subject to the constraint of equation (1.1).

We now find the stationary point of "$V^{\#}$" with respect to the variations in $u(s), x(s)$, $\lambda(s)$, and $x(T)$. Carrying out the variational problem and using the notation of Appendix "A" along with the variational notation (8, Chp.2) we get for stationary "V" (ie $\&V = 0$)

$$\int_t^T \Bigg[L_x(s;T,x(s),u(s),w(s))\,\&x(s) + L_u(s;T,x(s),u(s),w(s))\&u(s) + \&\lambda^T(s)\left[\frac{dx(s)}{ds} - f(s;x(s),u(s),w(s))\right] + \lambda^T(s)\left[\&\frac{dx(s)}{ds} - f_x(s;x(s),u(s),w(s))\&x - f_u(s;x(s),u(s),w(s))\&u(s)\right]\Bigg] ds \;+$$

$$L_x(t;T,x(t),u(t),w(t))\&x(t) - \lambda^T(t) f_x(t;x(t),u(t),w(t))\&x(t) + \mathcal{V}_x(T,x(T))\&x(T) \quad \ldots\ldots(1.7)$$

We now consider the term

$\int_t^T \lambda^T(s)\ \& \frac{dx(s)}{ds}\,ds$ and integrating by parts we get (1.8)

$$\lambda(s)\&x\Big|_t^T - \int_t^T \&x(s)\,\frac{d\lambda(s)}{ds}\,ds \quad \ldots\ldots(1.8)$$

substitute equation (1.8) back into the equation (1.7) and collecting terms in the variations of $\&x(s)$, $\&u(s)$, and $\&\lambda(s)$ also take the transpose of the complete equation to get (1.9).

$$\int_t^T \Bigg[\&x(s)^T\left[L_x(s;T,x(s),u(s),w(s)) - f_x^T(s;x(s),u(s),w(s))\,\lambda(s) - \frac{d\lambda(s)}{ds}\right] + \&u(s)^T\left[L_u(s;T,x(s),u(s),w(s)) - f_u^T(s;x(s),u(s),w(s))\,\lambda(s)\right] + \&\lambda(s)\left[\frac{dx(s)}{ds} - f^T(s;x(s),u(s),w(s))\right]\Bigg] ds \;+\; \&x(t)\Big[$$

$$L_x(t;T,x(t),u(t),w(t)) - \lambda(t) - f_x^T(t;x(t),u(t),w(t))\,\lambda(t)\Big] + \delta x(T)^T \Big[$$
$$\nu_x(T,x(T)^T\ \nu_x(T,x(T)) + \lambda(T)\Big] \quad \text{.........................(1.9)}$$

All the variables under the integral are not constrained and therefore their differentials are independent, equating their coefficients to zero one obtains the following Euler Lagrange Equations

$$\frac{dx(s)}{ds} = f(s;x(s),u(s),w(s))$$

$$\frac{d\lambda(s)}{ds} = L_x(s;T,x(s),w(s)) - f_x^T(s;x(s),u(s),w(s))\,\lambda(s)$$

$$f_u^T(s;x(s),u(s),w(s))\,\lambda(s) = L_u(s;T,x(s),u(s),w(s))$$

for all $t \leq s \leq T$ ....(1.10)

The Euler Equations are the basic necessary condition.

We now consider the boundary value terms (i.e. the last terms of (1.9)

$$\delta x(t)\Big[L_x(t;T,x(t),u(t),w(t)) - \lambda(t) - f_x^T(t;x(t),u(t),w(t))\,\lambda(t)\Big]$$
$$+\ \delta x(T)\Big[\nu_x(T,x(T)) + \lambda(T)\Big] \quad \text{........................(1.11)}$$

Since $x(t)$ is the current system state and cannot be changed by application of $u(t)$ it follows that all $\delta x(t)$ must be zero. Similarly $\delta x_i(T)$ $(i = 1,2, \ldots\ldots,n)$ are not all independent because of the constraint of equation (1.4).

Taking the differential of (1.4) one obtains (1.12)

$dg(T,x(T)) = g_x(T,x(T))\delta x(T) = 0$ ....(1.12)

if T is fixed.

If $g_i(T,x(T)) = 0$ $(i = 1,2, \ldots\ldots\ldots,n_g)$ are mutually consistent and non-redundant conditions "g_x" can be partitioned into a non-simgular $n_g x n_g$ matrix "$\hat{g}_x$" and a remainder "$\check{g}_x$".

The corresponding n_g components od dx(T) are called $d\hat{x}(T)$ and the remainder denoted by $d\check{x}(T)$. Thus equation (1.12) will become (1.13)

$$\hat{g}_x d\hat{x}(T) + \check{g}_x d\check{x}(T) = 0 \text{ ....(1.13)}$$

solving for $d\hat{x}(T)$ we get (1.14)

$$d\hat{x}(T) = -\hat{g}_x^{-1}\,\check{g}_x\,d\check{x}(T) \text{ ....(1.14)}$$

The last terms of (1.11) becomes

$$\left[\check{\lambda}(T) + \mathcal{V}_{\check{x}}(T,x(T)) - \check{g}_x^{T}\,\hat{g}_x^{-1\,T}\left[\hat{\lambda}(T) + \mathcal{V}_{\check{x}}(T,x(T))\right]\right]d\check{x}(T) \text{ ......(1.15)}$$

where $\hat{\lambda}_i(T)$ and $\mathcal{V}_{\hat{x}_i}$ are the components corresponding to $\&\hat{x}(T)$ whereas $\check{\lambda}_i(T)$ and $\mathcal{V}_{\check{x}_i}$ are those components corresponding to $\&\check{x}(T)$.

Now $d\check{x}_i(T)$ $(i = n_{g+1}, n_{g+2}, \ldots\ldots, n)$ are independent and the vanishing of their coefficients requires that

$$\check{\lambda}(T) + \mathcal{V}_{\check{x}}(T,x(T)) = \check{g}_x^{T}\,\hat{g}_x^{-1\,T}\left[\hat{\lambda}(T) + \mathcal{V}_{\hat{x}}(T,x(T))\right] \text{........(1.16)}$$

This set of $n-n_g$ conditions on $\lambda_i(T)$ or $x_i(T)$ $(i = 1,2, \ldots\ldots, n)$, along with the n_g conditions (1.4) completely specify the end conditions.

If $n_g=n$ equation (1.4) uniquely specifies x(T) so that none of the $\&x_i(T)$ $(i = 1,2, \ldots\ldots, n)$ are independent.

If $n_g=0$ x(T) is completely free and then

$$\lambda(T) + \mathcal{V}_x(T,x(T)) = 0 \text{ ....(1.17)}$$

These are the so called natural boundary conditions.

If $\mathcal{V}(T, xT)$ is independent of x(T), then

$$\lambda(T) = 0$$

Addi ional Necessary Conditions

Additionsl necessary conditions for the optimal control problem can be obtained by considering the classical work of Weierstrass and Legendre.

Forming the well-known Weierstrass E- function

$$E(x,u,u^o,s) = L(x,u,s) - L(x,u^o,s) - \langle u - u^o, L_u(x,u^o,s)\rangle \quad \ldots\ldots(1.18)$$

and by the Weierstrass necessary condition (see 30) it must be the case that $E(x,u,u^o,s) \geq o$. It is clear by inspection that Eis the quadratic remainder in the Taylor series of L at $u = u^o$. Using the well-known estimate for the remainder we have

$$E(x,u,u^o,s) = \|u - u^o\|^2_{L_{uu}(x,u + \theta(u - u^o),s)} \quad \ldots\ldots\ldots\ldots(1.19)$$

where θ is $0 \leq \theta \leq 1$

which is nonnegative definite if $L_{uu} > 0$. This is just Legendre's necessary condition.

Note : $\langle x,y\rangle$ is the well known inner product i.e., $\langle x,y\rangle = x^T y$.

SECTION I

PART II ADDITIONAL NECESSARY CONDITIONS AND SUFFICIENCY CONDITIONS

Another approach to the problem is to use Classical Hamilton-Jacobi Theory as applied to Calculus of Variations.

Hamilton and Jacobi recognized that the relationship of reducing a first order partial differential equations into a system of associated ordinary differential equations could be reversed. In general of course the integration of a partial differential equation is usually a more difficult problem than that of solving an ordinary differential equation. In mathematical physics and mechanics one is often led, however, to a system of ordinary differential equations in canonical form. These equations may be difficult to integrate by elementary methods while the corresponding partial differential equation is manageable.

Eulers differential equations in canonical form can be placed in a suitable form by use of Legendre Transformation and its inverse to obtain an appropriate partial differential equation.

Let $$J(x(s),\dot{x}(s),u(s),s) = L(x(s),u(s),s) + \lambda^T \left(\frac{dx}{ds} - f(s;x,u) \right) \quad \text{..........(2.1)}$$

The resulting Euler equations for the above functional are

$$\frac{d}{ds}J_{\dot{x}i} - J_{x_i} = 0 \qquad (i \quad 1,2,\ldots\ldots,n) \qquad \text{...........(2.2)}$$

We now replace the variational problem by an equivalent canonical variational problem which leads to a system of 2n canonical differential equations. For this purpose we introduce the moments

$\lambda = J_{\dot{x}}$ and assume that $\dot{x}$ can be calculated from this equation in an appropriate domain $(\dot{x},x,u,s)$ This condition is true if

$$\left(J_{\dot{x}_j \dot{x}_i} \right) \neq 0$$

We now introduce the inverse function $\mathcal{H}(\lambda,x,u,s)$ under the Legendre transformation where x,u,s are the untransformed parameters.

From the Legendre transformation it must be true that

$$= J_{\dot{x}}(\dot{x},x,u,s), \qquad \dot{x} = \mathcal{H}(\lambda,x,u,s)$$

and ....(2.3)

$$J(\dot{x},x,u,s) + \mathcal{H}(\lambda,x,u,s) = \langle \dot{x},\lambda \rangle$$

Differentiating equation (2.3) with respect to "x" we obtain

$$J_x(\dot{x},x,u,s) + \mathcal{H}_x(\lambda,x,u,s) = 0 \qquad \text{...........(2.4)}$$

From which we immediately obtain the following canonical form for the Euler equations

$$\dot{x} = \mathcal{H}_\lambda(\lambda,x,u,s) \qquad \text{.....................(2.5)}$$
$$\dot{\lambda} = J_x = -\mathcal{H}_x(\lambda,x,u,s)$$

In turn these equations must satisfy the Hamilton Jacobi equation

$$V_s + \mathcal{H}(V_x,u,x,s) = 0 \qquad \text{.....................(2.6)}$$

Replacing the value for $J(\dot{x},x,u,s)$ we see that canonical form becomes

$$\frac{d\varsigma(s)}{ds} = f(s;x(s),u(s)) \qquad \text{.....................(2.7)}$$

$$\frac{d\lambda}{ds}(s) = L_x(s;x(s),u(s)) - f_x^T(s;x(s),u(s))\,\lambda(s)$$

and the Hamilton$^{\text{ian}}$ becomes

$$\mathcal{H}(V_x, x, u, s) = -L(s; x(s), u(s)) + \langle \dot{x}, \lambda(s) \rangle \quad \ldots\ldots(2.8)$$

If we let $p(s) = -\lambda(s)$ the Hamilton becomes the negative of the so-called pre-Hamiltonian in Pontryagin's Maximum Principle (see).

The pre-Hamiltonian is given by

$$\mathcal{H}^*(V_x, x, u, s) = L(x, u; s) + \langle \dot{x}, \xi(s) \rangle$$

Controllability and Observability

The concepts of controllability and observability where developed by R. E. Kalman, L. Markus and E. B. Lee with the majority of the work contributed by R. E. Kalman. These concepts are in their infancy stages and a great deal of work is still required to complete the study, as yet the concepts apply only to linear systems except for the Local Controllability Theorem. These concepts are sufficiency conditions and for this reason do not alone determine whether or not a system is controllable or observable.

Definition: A state x(s) is said to be controllable at time t_o if there exists a control function $u^1(s)$, depending on x and t_o and defined over some finite closed interval $[t_o, T]$, such that $\phi_2(T; x, t_o) = 0$ (for terminal point at the origin). If this is true for every state x, we say that the plant is completely controllable at time t_o; if this is true for every t_o, we simply say that the plant is completely controllable.

This definition can also be stated in terms of differential equationsas follows.

A linear system (B-6,7) is completely controllable at time t_o if it is not algebraically equivalent, for all $\left[s \quad t_o\right]$, to a system of the type

$$\frac{dx'}{ds} = F^{11}(s)x' + F^{12}(s)x^2 + G^1(s)u(s)$$

$$\frac{dx^2}{ds} = F^{22}(s)x^2 \qquad \qquad ....(2.9)$$

$$y(s) = H^1(s)x^1(s) + H^2(s)x^2(s)$$

where the vectors x^1 and x^2 are subgroups of the state vector x and contain n_I and $n_2 = (n - n_I)$ components respectively.

In other words, it is not possible to find a coordinate system in which the state variables x_i are separated into two groups, $x' = (x_I, .., x_{n_I})$ and $x^2 = (x_{nI}, ..., x_n)$, such that the second group is not affected either by the first group or by the inputs to the system. If one could find such a coordinate system, we would have the state of affairs depicted schematically in Fig. (I-2.1)

Controllability is a system property, like stability, which is completely independent of the way in which the outputs of the system are formed. It is a property of the couple $\left[F(s), G(s)\right]$

Definition: Observable Costates,- A vector x# in the dual space is called a costate. The value of the real linear function x# at x is $\left[x\#, x\right]$.

A costate $x\#_o$ of (B-6,7), with $u(s) \equiv 0$, is observable at time t_o if there exists a linear function $u\#(s)$ defined on some interval $\left[T_{-I}, t_o\right]$ and depending in general on $x\#_o$ and t_o, such that

$$\left[x\#_o, x(t_o)\right] = \int_{T_{-I}}^{t_o} \left[u\#(s), y(s)\right] ds \quad \ldots\ldots\ldots\ldots(2.10)$$

In other words, a costate observable at t_o is a linear function whose value can be explicitly evaluated (no matter what the actual state of the system happens to be) by means of a linear operation on the output of the system during a finite interval preceding t_o.

A state x_o is unobservable at time t_o whenever $\left[x^{\#}, x_o\right] = 0$ for all $x^{\#}$.

We can also define observability in a slightly different manner as follows. A linear dynamical system (B-6,7) is completely observable at time t_o if it is not algebraically equivalent, for all $s = t_o$, to any system of the type

$$\frac{dx^1}{ds} = F^{11}(s)x^1 + G^1(s)u(s)$$

$$\frac{dx^2}{ds} = F^{21}(s)x^1 + F^{22}(s)x^2 + G^2(s)u(s)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(2.11)$$

$$y(s) = H^1(s)x^1(s)$$

This state of affairs, if it existed, is depicted in Fig. (I-2.2).

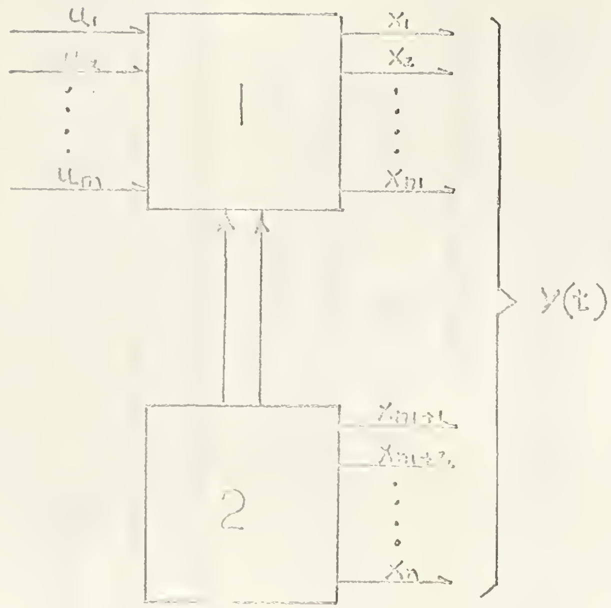


FIG I (2-1)

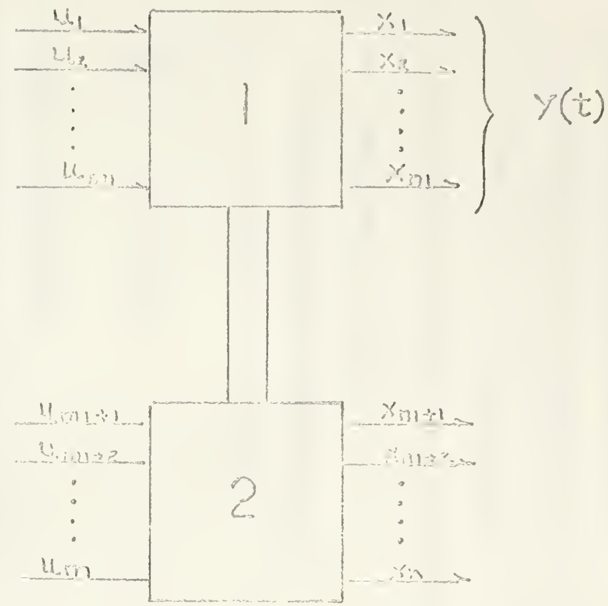


FIG I (2-2)

Motivation for Controllability

Consider the problem where the loss function $L(x,u,s)$ is given by $1/2 \; || \, u(s) \, ||^2$. This problem is sometimes called the minimum energy problem because the integral of this particular loss function is a measure of energy.

The resulting Euler equations are

$$\frac{dx}{ds} = F(s)x(s) + G(s)u(s)$$

$$\frac{d\lambda}{ds} = -F^T(s)\lambda(s) \qquad \text{.....................(2.12)}$$

$$u(s) = G^T(s)\lambda(s)$$

The solution for $\lambda(s)$ is given by

$$\lambda(s;\xi_o,t_o) = \Lambda(s,t_o)\,\xi_o \qquad \text{.................(2.13)}$$

where $\Psi(s,t_o)$ satisfy

$$\frac{d\Delta}{ds} = -F^T(s)\Lambda(s,t_o) \qquad \text{.................(2.14)}$$

From Appendix (B) we see that

$$\Lambda(s,t_o) = \Phi^T(t_o,s) \qquad \text{.................(2.15)}$$

Using these results the control law now becomes

$$u(s) = G^T\Phi^T(t_o,s)\,\xi_o$$

We have yet to establish a relationship between ξ_o and x_o. To do this consider the solution for the system using equation (B-10).

$$\varphi(s;x_o,t_o) = \Phi(s,t_o)x_o + \int_{t_o}^{s}\Phi(s,\sigma)G(\sigma)G^T(\sigma)\Phi^T(t_o,\sigma)\,\xi_o\,d\sigma$$

Define $\qquad$ ....(2.16)

$$W(t_o,s) = \int_{t_o}^{s}\Phi(t_o,\sigma)G^T(\sigma)\Phi^T(t_o,\sigma)\,d\sigma$$

and evaluate equation (2.16) at s = T.

$$\varphi(T;x_o,t_o) = \Phi(T,t_o)x_o + \Phi(T,t_o)W(t_o,T)\xi_o \quad \text{..........(2.17)}$$

Solving equation (2.17) for ξ_o to get

$$\Phi(t_o,T)x(T) - x_o = W(t_o,T)\xi_0 \quad \text{...............(2.18)}$$

Using this result we now see the control law becomes

$$u(s) = -G^T(s)\Phi^T(t_o,s)W^{-1}(t_o,T)\left[x_o - \Phi(t_o,T)x(T)\right] \quad \text{..............(2.19)}$$

From which we find that

$$\int_{t_o}^{T}\left\| u(s)\right\|^2 ds = \left\| x_o - \Phi(t_o,T)x(T)\right\|^2_{W^{-1}(t_o,T)}$$

From which we see that for the energy to be positive (which it must be if the system is to have physical meaning) it must be the case that $W(t_o,T)$ be positive definite. The extension of this idea to the general case is done by hypothesis, that is to say we assume that the above condition holds true in general and proof that this is the case.

To determine controllability of a system the following lemma can be used.

Lemma 1: A plant is completely controllable at time s(i) if and (ii) only if the symmetric matrix

$$W(t_o,T) = \int_{t_o}^{T}\Phi(t_o,s)G(s)G^T(s)\Phi^T(t_o s)ds \quad \text{........(2.20)}$$

is positive definite for some $T > t_o$.

An equivalent statement is that $W(t_o,T)$ must satisfy the following differential equation.

$$\frac{dW(t_o,T)}{dt_o} = F(t_o)W + WF^T(t_o) - G(t_o)G^T(t_o)$$

subject to the condition that $W(T) = 0$

Proof: (i) set $u'(s) = -G^T(t)\,\Phi^T(t_o,s)W^{-1}(t_o,T)x(t_o)$ ....(2.21)

and then substitute this into the solution of the plant given by equations (B-6,7) with a solution given by equation (B-10). It is desirable to return some system to the origin by a control function $u'(t)$ subject to some performance criterion. Hence, if upon substitution of $u'(s)$ into the solution of the plant equation the state is returned to the origin the Lemma is proved for (i). Upon substitution

$$\varphi_u(T;x_0,t_o) = \Phi(T,t_o)x_o - \int_{t_o}^{T}\Phi(T,\xi)G(\xi)G^T(\xi)\Phi^T(t_o,\xi)W^{-}(t_o,T)\,x(t_o)d\xi \quad \text{.........(2.22)}$$

By the lemma and from the fact that $\Phi(s,\xi) = \Phi(s,t_o)\,\Phi(t_o,\xi)$ equation (2.22) becomes

$$\varphi_u(T;x_o,t_o) - \Phi(T,t_o)x_o - \Phi(T,t_o)x_o$$

Hence $\varphi_u(T;x_o,t_o) = 0$

(ii) We will prove this part by contradiction. By the lemma it is required that the matrix $W(t_o,T)$ be positive definite.

Assume that there exists an $x \neq 0$ such that $\|x\|^2_{W(t_o,T)} = 0$

Define:

$$u^2(s) = -G^T(s)\,\Phi^T(t_o,s)x(t_o)$$

which implies that

$$\|x(t_o)\|^2_{W(t_o,T)} = \int_{t_o}^{T}\left[u2(s)\right]^2 ds = 0$$

Since $u^2(s)$ is continuous in s, it therefore identically zero in the interval $\left[t_o, T\right]$.

On the other hand, if the plant is completely controllable at t_o, there exists a control function $u'(s)$ from (i) which satisfies the relation

$$x(t_o) = -\int_{t_o}^{T} \Phi(t_o,s)G(s)u'(s)ds \qquad \text{.......................(2.23)}$$

To see that this relationship holds just substitute the value for $u^1(s)$ given by equation (2.21) into equation (2.23) to get the result $x(t_o) = x(t_o)$.

Therefore it must be the case that

$$\left\| x(t_o) \right\|^2 - \int_{t_o}^{T} \left[u^1(s), u^2(s) \right] ds = 0$$

contradicting the assumption that $x(t_o) \neq 0$ Q.E.D.

The last part is proved by differentiating equation (2.20) with respect to "t_o".

$$\frac{dW(t_o,T)}{dt_o} = \int_{t_o}^{T} \frac{\partial}{\partial t_o} \Phi(t_o,s) \left[G(s)G^T(s)\Phi^T(t_o,s) \right] ds + \int_{t_o}^{T} \Phi(t_o,s)G(s)G^T(s)\frac{\partial}{\partial t_o}\Phi^T(t_o,s)ds - G(t_o)G^T(t_o) \qquad \text{...........(2.24)}$$

and from the equations $\dfrac{d\Phi(t_o,s)}{dt_o} = F(t_o)\Phi(t_o,s)$

and $\dfrac{d\Phi^T(t_o,s)}{dt_o} = \Phi^T(t_o,s)F^T(t_o)$ we get

$$\frac{dW(t_o,T)}{dt_o} = F(t_o)W(t_o,T) + W(t_o,T)F^T(t_o) - G(t_o)G^T(t_o)$$

and this completes the proof.

The proofs given for controllabilty are applicable when the terminal point of the system is the origin. It follows easily by a slight extension of the preceding arguments that z(s) (where z(s) is some desired state at s = T) is reachable from x(i.e., there exists a motion φ which meets x at t_o and z at T) if and only if the equation

$$x(t_o) - \Phi(t_o,T)z = W(t_o,T)v(t_o) \quad \ldots\ldots\ldots\ldots(2.25)$$

has a solution, in which case

$$u^1(s) = -G^T(s)\,\Phi^T(t_o,s)v(t_o) \quad \ldots\ldots\ldots\ldots(2.26)$$

is the appropriate control function.

To show this substitute $u^1(s)$ into equation (B-10) to get

$$\varphi_{u^1}(T;x_o,t_o) = \Phi(T,t_o)x_o - \int_{t_o}^{T} \Phi(T,\xi)G(\xi)G^T(\xi)\,\Phi^T(t_o\,\xi)v(t_o)d\xi$$

which becomes from lemma (I).

$$\varphi_{u'}(T;x_o,t_o) = \Phi(T,t_o)x_o - W(t_o,T)v(t_o) \quad \ldots\ldots\ldots\ldots(2.27)$$

Assuming from the lemma that $W(t_o,T)$ is positive definite we can solve for $v(t_o)$ from equation (2.25) to get

$$v(t_o) = W^{-1}(t_o,T)\left[\,x(t_o) - \Phi(t_o,T)z(T)\right]$$

which in turn is substituted into equation (2.27).

Hence

$$\begin{aligned}\varphi_{u'}(T;x_o,t_o) &= \Phi(T,t_o)x_o - W(t_o,T)W^{-1}(t_o,T)\left[x_o - \Phi(t_o,T)z\right]\\ &= \Phi(t_o,T)z(T)\end{aligned}$$

The above conditions are applicable for linear time varying systems, there is a somewhat more convenient form for constant linear systems. In the case of constant systems the notions of complete controllability and complete observability do not depend on the choice of t_o. Lemma (2): A constant plant is completely controllable (i) if and (ii) only if

$$\text{rank} \quad \left[G, FG, \ldots\ldots, F^{n-I}G\right] = n \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.28)$$

Proof: (i) By the previous lemma it is sufficient to show that $W(0,T)$ is positive definite no matter how small $T - t_o > 0$.

Let $g' \ldots\ldots, g^m$ be the columns of G. If $W(0,T)$ is positive semidefinite, then proceeding as in part (ii) of the previous lemma we conclude that there is a vector $x(0) \neq 0$ such that $\|x\|^2_{W(0,T)} = 0.$

Define

$$u^2(s) = -G^T \Phi^T(s)x$$

which implies that

$$\|x\|^2_{W(0,T)} = \int_0^T \left[u^2(s)\right]^2 ds = 0$$

and since $u^2(s)$ is continuous in s, it is identically zero in the interval $[0,T]$.

Therefore in the case of constant coefficients,

$$u^2(s) = -G^T \exp F^T s \; x = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2.29)$$

and therefore,

$$\left[x, \exp Fs \; g^i\right] = 0 \quad \text{for } i = I, \ldots\ldots\ldots, m$$

where $(g^1, g_2, \ldots gm)$ are the columns of G

Differentiating j times with respect to s, and then setting s = 0, we get

$$\left[x,\ F^j g^i \right] = 0 \quad \text{for all } i = I, \ldots\ldots, m$$
$$\text{and} \quad j = I, \ldots\ldots, n-I$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.30)$$

If equation (2.28) holds, this implies that x is orthogonal to set of generators of E^n, contradicting the assumption that $x \neq 0$.

(ii) Assume the plant is completely controllable but (2.28) is false. Then there is a vector $x \neq 0$ which satisfies (2.29). By the Cayley - Hamilton theorem

$$\left[x,\ \exp Fs\ g^i \right] = \left[x, \left(\sum_{j=0}^{\infty} (Fs)^j/j! \right) g^i \right] = \left[x, \left(\sum_{j=0}^{m-I} \alpha_j (Fs)^j \right) g^i \right] = 0$$

where i = I, ...m

It follows that $\| x \|^2_{W(O,T)} = 0$ for all T, contradicting the assumption of complete controllability. Q.E.D.

To determine observability the following lemma is used.

Lemma 3: A plant is completely observable (i) if and (ii) only if the symmetric matrix

$$M(t_o, T_{-I}) = \int_{T-I}^{t_o} \Phi^T(s,t_o) H^T(s) H(s)\, \Phi(s,t_o)\, ds \quad \ldots.(2.31)$$

is positive definite for all t_o.

This idea can also be extended to linear constant systems under the following lemma.

Lemma 4: A constant plant is completely observable (i) if and (ii) only if

$$\text{rank} \left[H^T, F^T H^T, \ldots\ldots\ldots, (F^T)^{n-I} H^T \right] = n$$

Proof: The proof of these lemmas is achieved indirectly by showing that all theorems on controllability can be "dualized" to yield analogous results on observability.

Duality relations

(i) $\quad s - t_o = t_o - t$

(ii) $\quad F^*(t) = F^T(s)\,, \quad G^*(t) = H^T(s)\,, \quad H^*(t) = G^T(s)$

(iii) $\quad \Phi^*(t, t_o) = \Phi^T(t_o, s)$

where the dual system is given by

$$\frac{dx^*}{dt} = F^*(t)x^* + G^*(t)u^*(t)$$

$$y^*(t) = H^*(t)x^*(t)$$

Hence it can be stated that a plant is completely controllable if, and only if, its dual is completely observable, and conversely.

Definition: A plant is uniformly completely controllable if the following relations hold for all s:

(i) $\quad 0 < \alpha_o(\sigma)I \leqq W(s, s + \sigma) \leqq \alpha_1(\sigma)I$

(ii) $\quad 0 < B_o(\sigma)I \leqq (s + \sigma, s)W(s, s + \sigma)\Phi^T(s + \sigma, s) \leqq B(\sigma)I$

where σ is a fixed constant. In other words, one can always transfer x to 0 and 0 to x in a finite length σ of time; moreover, such a transfer can never take place using an arbitrarily small amount (or requiring an arbitrarily large amount) of control energy.

Similarly we can define uniformly complete observability by use of the duality relations. (i.e., if a plant is uniformly completely controllable its dual is uniformly completely observable).

The conditions of uniform complete controllability and observability will be used in a later section to prove a stability theorem for a linear system.

By using the Gronwall-Bellman lemma:

$$\int_{t_1}^{t_2} \| F(\tau) \| \, d\tau \leq \gamma \ (t_2 - t_1) \text{ for all } t_1, \text{ and } t_2$$

We arrive at the condition that

$$\| \Phi(s,\tau) \| \leq \alpha_3(| s \ - \tau |) \quad \text{for all } s, \tau \ \ \ldots\ldots(2.32)$$

The ideas of controllability is extended to the nonlinear equation

$$\frac{dx}{ds} = f(s;x,u) \text{ where } \quad f(s;0,0) = 0 \quad \ldots\ldots\ldots\ldots(2.33)$$

Linearization of f(s;x,u,) results in the equation (B-6,7) where

$$F(s) = \frac{\partial f^i(s;0,0)}{\partial xj} \quad \text{and} \quad G(s) = \frac{\partial f^i(s;0,0)}{\partial uj}$$

Local Controllability Theorem

The nonlinear equation (2.33) is locally completely controllable at the origin at time t_o if (B-6,7) is completely controllable at time t_o.

Proof: Assume that a solution for equation (2.33) exists, (Note, it can be shown that under certain conditions a solution to (2.33) does exist but this is beyond the scope of this paper.).

We let u_ξ be the function

$$u(s,\xi) = + \ G^T(s)\Phi^T(t_o,s)\xi \ \ldots\ldots\ldots\ldots\ldots\ldots(2.34)$$

From the solution to the equation (B-6) we see that,

$$\phi(T;x_o,t_o,u_\xi) = \Phi(T,t_o)x_o + \Phi(T,t_o)W(t_o,T) \qquad \ldots\ldots(2.35)$$

so the Jacobian is

$$\left[\partial \Phi^{i}(T;x_o,t_o,u_\xi)/_{\partial \xi^{j}}\right] = \Phi(T,t_o)W(t_o,T)$$

which is nonsingular.

Hence by implicit function theorem we can solve for u_ξ in term of x_o, denote as

$$u_\xi = \Psi(x_o)$$

Further it must be the case that $\Phi(T;x_o,t_o u) = 0$ which is the requirement that the plant must obey.

SECTION II

PART I SOLUTION OF THE TWO POINT BOUNDARY VALUE PROBLEM FOR A CONSTANT COEFFICIENT LINEAR SYSTEM

In the case of constant coefficient linear systems the plant can be described by the following differential equations

$$\frac{dx(s)}{ds} = Fx(s) + Gu(s) + Kw(s) \quad \ldots\ldots\ldots\ldots\ldots\ldots(1.1)$$

where F is a constant n x n matrix

G is a constant n x m matrix

K is a constant n x r matrix

The output of the system is expressed by the equation

$$y(s) = Hx(s) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.2)$$

The problem of a linear system with constant coefficients has been considered in detail by methods other than presented here, most of these methods employ Laplace transforms and Wiener's Spectorial Factorization techniques.

The performance criterion to be considered is of the form

$$V = \lim_{T \to \infty} \int_t^T L(x(s),u(s))ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.3)$$

subject to the general end point boundary conditions of the form

$$Ax(T) + B\lambda(T) = d \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.4)$$

where A and B are constant matrices.

Further we assume that the loss function L(x(s),u(s)) is a quadratic function of x(s),u(s), and z(s) with constant coefficients and T is fixed.

The desired output of the system is denoted by z(s) and is included in the loss function.

Let us consider the quadratic loss function of the form

$$L(x(s),u(s)) = \tfrac{1}{2}\left\{ \|y(s) - z(s)\|_Q^2 + \|u(s)\|_R^2 \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.5)$$

upon substitution of equation (1.2) into equation (1.5) we get the loss function

$$L(x(s),u(s)) = \tfrac{1}{2}\left\{ \| Hx(s) - z(s)\|_Q^2 + \|u(s)\|_R^2 \right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.6)$$

where Q and R are symmetric positive definite constant matrices.

With the above restrictions imposed on the system the Euler equations become

$$\frac{dx(s)}{ds} = Fx(s) + Gu(s) + Kw(s)$$

$$\frac{d\lambda(s)}{ds} = H^TQHx(s) - F^T\lambda(s) - H^TQz(s)$$

$$Ru(s) = G^T\lambda(s) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.7)$$

because R is nonsingular we can solve for u(s) to get the linear control law

$$u(s) = R^{-1}G^T\lambda(s) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.8)$$

Now substitution of equation (1.8) back into the Euler equations results in

$$\frac{dx(s)}{ds} = Fx(s) + GR^{-1}G^T\lambda(s) + Kw(s)$$

$$\frac{d\lambda(s)}{ds} = H^TQHx(s) - F^T\lambda(s) - H^TQz(s) \quad \ldots\ldots\ldots (1.9)$$

rewriting the above equations in the following form

$$\begin{pmatrix} \dfrac{dx(s)}{ds} \\ \dfrac{d\lambda(s)}{ds} \end{pmatrix} = \begin{pmatrix} F & GR^{-1}G^T \\ H^T QH & -F^T \end{pmatrix} \begin{pmatrix} x(s) \\ \lambda(s) \end{pmatrix} + \begin{pmatrix} Kw(s) \\ -H^T Qz(s) \end{pmatrix} \quad \ldots\ldots\ldots\ldots(1.10)$$

and diagonalize the matrix $\begin{pmatrix} F & GR^{-1}G^T \\ H^T QH & -F^T \end{pmatrix}$ under the unitary transformation U such that

$$U \begin{pmatrix} F & GR^{-1}G^T \\ H^T QH & -F^T \end{pmatrix} U^{-1} = D$$

If we investigate the constant matrix $\begin{pmatrix} F & GR^{-1}G^T \\ H^T QH & -F^T \end{pmatrix}$ we find that the right half of the "s" plane (Note; s here is the Laplace transform) has "n" roots whose real part has an opposite sign in the left-hand side of the "s" plane. Using this fact that the roots are symmetrically located about the imaginary axis in the s plane we subdivide the diagonalized matrix D into two parts D_r for roots with positive real parts and D_l for roots with negative real parts. Further it must be the case that $D_r = -D_l$.

In cases of roots on the imaginary axis we assume that they have a real part of ε.

The fundamental solution for D_r and D_l will be designated as $\Phi_l(t,t_o)$ and $\Phi_r(t,t_o)$. They must in turn obey the equations

$$\frac{d\Phi_l(t,t_o)}{dt} = D_l \Phi_l(t,t_o)$$

$$\frac{d\Phi_r(t,t_o)}{dt} = D_r \Phi_r(t,t_o)$$

where $\Phi_1(t,t_o)$ decays to zero, where $\Phi_r(t,t_o)$ grows unbounded as $t \longrightarrow \infty$

Under the transformation of the matrix U to diagonalize the matrix

$\begin{pmatrix} F & GR\,G^T \\ H^TQH & -\,F^T \end{pmatrix}$ we now define a new set of variables $\begin{pmatrix} \xi(s) \\ \eta(s) \end{pmatrix} = U \begin{pmatrix} x(s) \\ \lambda(s) \end{pmatrix}$

Thus equation (1.10) becomes upon substitution of the new variables

$$\begin{pmatrix} \frac{dx(s)}{ds} \\ \frac{d\lambda(s)}{ds} \end{pmatrix} = U^{-1}\begin{pmatrix} \frac{d\xi(s)}{ds} \\ \frac{d\eta(s)}{ds} \end{pmatrix} = \begin{pmatrix} F & GR\,G^T \\ H^TQH & -F^T \end{pmatrix} U^{-1} \begin{pmatrix} \xi(s) \\ \eta(s) \end{pmatrix} + \begin{pmatrix} Kw(s) \\ -H^TQz(s) \end{pmatrix}$$

we now multiply both sides by U and using the fact that $UU^{-1} = I$ we get

$$\begin{pmatrix} \frac{d\xi(s)}{ds} \\ \frac{d\eta(s)}{ds} \end{pmatrix} = \begin{pmatrix} D_r & 0 \\ D & D_1 \end{pmatrix} \begin{pmatrix} \xi(s) \\ \eta(s) \end{pmatrix} + U \begin{pmatrix} \hat{w}(s) \\ \check{w}(s) \end{pmatrix}$$

where $\hat{w}(s) = Kw(s)$, and $\check{w}(s) = -H^TQz(s)$.

The solution of these equations at $s = T$ is given by

$$\xi(T) = \Phi_r(T,t)\,\xi(t) + \int_t^T \Phi_r(T,\sigma)w^*(\sigma)\,d\sigma \quad \text{.........(1.11)}$$

$$\eta(T) = \Phi_1(T,t)\,\eta(t) + \int_t^T \Phi_1(T,\sigma)w^*(\sigma)\,d\sigma$$

In terms of the new variables, and partitioning U into $\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ whose inverse is given by $\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$ the boundary conditions become

$$(AV_{11} + BV_{21})\,\xi(T) \;+\; (AV_{12} + BV_{22})\,\eta(T) = d \quad \ldots\ldots\ldots\ldots(1.12)$$

substitution of equation (1.11) into equation (1.12) to get

$$(AV_{11} + BV_{21})\,\Phi_r(T,t)\,\xi(t) \;+\; (AV_{12} + BV_{22})\,\Phi_1(T,t)\,\eta(t)$$

$$\int_t^T \Phi_r(T,\sigma)\hat{w}^*(\sigma)\,d\sigma \;+\; \int_t^T \Phi_1(T,\sigma)w^*(\sigma)\,d\sigma = d \quad \ldots\ldots\ldots\ldots(1.13)$$

Further returning to the original variables i.e.,

$$\begin{pmatrix} \xi(s) \\ \eta(s) \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} x(s) \\ \lambda(s) \end{pmatrix}$$ and equation (1.13) becomes

$$(AV_{11} + BV_{21})\,\Phi_r(T,t)\left[U_{11}\,x(t) + U_{21}\,\lambda(t)\right] + (AV_{12} + BV_{22}\Big|$$
$$\Phi_1(T,t)\left[U_{21}x(t) + U_{22}\,\lambda(t)\right]\Big) + \int_t^T \Phi_r(T,\sigma)\hat{w}^*(\sigma)\,d\sigma$$
$$\int_t^T \Phi_1(T,\sigma)\hat{w}^*(\sigma)\,d\sigma = d \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.14)$$

If we assume that the system is unforced and that we solve the regulator problem with the final state of $y(s) = 0$. Then the equation (1.14) becomes

$$\left[(AV_{11} + BV_{21})\,\Phi_r(T,t)U_{11} + (AV_{12} + BV_{22})\,\Phi_1(T,t)U_{21}\right]\;x(t)$$

$$\left[(AV_{11} + BV_{21})\,\Phi_r(T,t)U_{21} + (AV_{12} + BV_{22})\,\Phi_1(T,t)U_{22}\right]\;\lambda(t) = d$$

Thus from the control law we get

$u(t) = R^{-1}G^T\lambda(t)$ where $\lambda(t)$ is given by the following equation

$$\lambda(t) = \left[(AV_{11} + BV_{21})\,\Phi_r(T,t)U_{21} + (AV_{12} + BV_{22})\,\Phi_1(T,t)U_{22}\right]^{-1}$$

$$\left[d - \left[(AV_{11} + BV_{21})\,\Phi_r(T,t)U_{11} + (AV_{12} + BV_{22})\,\Phi_1(T,t)U_{21}\right] x(t)\right]$$

In the steady state case when $T \rightarrow \infty$ the equations reduce to

$$u(t) = R^{-1}G^TU_{12}^{-1}U_{11}\, x(t) \qquad \text{.................................(1.15)}$$

Stability Properties of Euler Equations for the Linear System

The symmetric properties of the matrices in the Euler equations imply the instability of these equations. It will be shown that no stable equilibrium point exists and that every equilibrium point is a saddle type singularity to which there converge "n" stable solutions and to which there diverge "n" other unstable solutions.

Consider the following Euler equations

$$\begin{pmatrix} \frac{dx(s)}{ds} \\ \frac{d\lambda(s)}{ds} \end{pmatrix} = \begin{pmatrix} F & GR^{-1}G^T \\ H^TQH & -F^T \end{pmatrix} \begin{pmatrix} x(s) \\ \lambda(s) \end{pmatrix} \qquad \text{................(1.16)}$$

let $R\# = GR^{-1}G^T$ and $Q\# = H^TQH$ results in

$$\begin{pmatrix} \frac{dx(s)}{ds} \\ \frac{d\lambda(s)}{ds} \end{pmatrix} = \begin{pmatrix} F & R\# \\ Q\# & -F^T \end{pmatrix} \begin{pmatrix} x(s) \\ \lambda(s) \end{pmatrix} \quad \text{or} \quad \begin{aligned} \frac{dx(s)}{ds} - Fx(s) &= R\#\lambda(s) \\ \frac{d\lambda(s)}{ds} + F^T\lambda(s) &= Q\#x(s) \end{aligned} \qquad \text{....(1.17)}$$

Taking the Laplace transform of the above equations we obtain

$$\begin{aligned} (sI - F)X(s) - R\#\Lambda(s) &= \text{const.} \\ (sI - F)\Lambda(s) - Q\#X(s) &= \text{const.} \end{aligned} \qquad \text{.............(1.18)}$$

rewrite as

$$\begin{aligned} (F - sI)X(s) + R\#\Lambda(s) &= \text{const.} \\ Q\#X(s) - (F^T + sI)\Lambda(s) &= \text{const.} \end{aligned} \qquad \text{.................(1.19)}$$

where the const. takes care of the initial conditions.

The stability of the equation will depend on the roots of the polynomial in the complex "s" plane

$$\begin{vmatrix} F-sI & R\# \\ Q\# & -F^T - sI \end{vmatrix}$$

where roots in the right-hand side correspond to unstable characteristic solution (growing exponentials), and roots in the left-hand plane correspond to stable characteristic solution (decaying exponential).

Since the determinant of a matrix equals the determinant of its transpose

$$\begin{vmatrix} F - sI & R\# \\ Q\# & -F^T - sI \end{vmatrix} = \begin{vmatrix} F^T - sI & Q\# \\ R\# & -F-sI \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots(1.20)$$

where use has been made of the fact that $Q\#$ and $R\#$ are symmetric.

By interchanging columns, then rows one obtains successively

$$\begin{vmatrix} F^T - sI & Q\# \\ R\# & -F - sI \end{vmatrix} = \begin{vmatrix} -Q\# & -P^T + sI \\ -F - sI & R\# \end{vmatrix} = \begin{vmatrix} F + sI & R\# \\ Q\# & -F^T + sI \end{vmatrix}$$

so that

$$\begin{vmatrix} F - sI & R\# \\ Q\# & -F^T - sI \end{vmatrix} = \begin{vmatrix} F - I(-s) & R\# \\ Q\# & -F^T - I(-s) \end{vmatrix} \ldots\ldots(1.21)$$

Thene, the roots must be located symmetrically about the imaginary axis in the "s" plane.

SECTION II

PART II SOLUTION OF THE TWO POINT BOUNDARY VALUE PROBLEM FOR LINEAR NON-CONSTANT SYSTEM

In the case where the coefficients are non-constant (i.e., function of "s") we can no longer employ the diagonalization technique used for the solution of the constant coefficient linear system. We can, however, develop a general method which can be solved analytically in some cases and by numerical techniques in general.

Consider the plant

$$\frac{dx(s)}{ds} = F(s)\,x(s) + G(s)\,u(s) + K(s)\,w(s)$$
$$y(s) = H(s)\,x(s) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.1)$$

subject to the boundary conditions

$$x(s) = x(s)$$
$$A\,x(T) + B\,\lambda(T) = d \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2.2)$$

Once again we let the loss function be quadratic and of the form

$$L(x(s),u(s)) = \frac{1}{2}\Big[\big(y(s)^T - z(s)^T\big)\,Q(s)\,\big(y(s) - z(s)\big) + \frac{1}{2}\,u(s)^T R(s)\,u(s)\Big] \ldots\ldots\ldots\ldots\ldots\ldots(2.3)$$

where Q(s) and R(s) are symmetric positive definite matries of class C^2 and z(s) is the desired output.

Forming the Euler equations and eliminating u(s) we obtain a canonical system of the form

$$\frac{dx(s)}{ds} = F(s)\,x(s) + R^{\#}(s)\,\lambda(s) + \hat{w}(s)$$
$$\frac{d\lambda(s)}{ds} = Q^{\#}(s)\,x(s) - F^T(s)\,\lambda(s) + \check{w}(s) \qquad \ldots\ldots\ldots\ldots(2.4)$$

where $\hat{w}(s) = K(s)\, w(s)$

$$\check{w}(s) = -H(s)^T Q(s)\, z(s) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.5)$$

and $R^{\#}(s) = G(s)R(s)G(s)^T$

$$Q^{\#}(s) = H(s)^T Q(s) H(s) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.6)$$

Rewriting the equations in the following form

$$\begin{pmatrix} \dfrac{dx(s)}{ds} \\ \dfrac{d\lambda(s)}{ds} \end{pmatrix} = \begin{pmatrix} F(s) & R^{\#}(s) \\ Q^{\#}(s) & -F^T(s) \end{pmatrix} \begin{pmatrix} x(s) \\ \lambda(s) \end{pmatrix} + \begin{pmatrix} \hat{w}(s) \\ \check{w}(s) \end{pmatrix} \ldots\ldots\ldots(2.7)$$

Let $\Phi(s,t)$ be the transition matrix for the above system and denote the matrix $\begin{pmatrix} F(s) & R^{\#}(s) \\ Q^{\#}(s) & -F^T(s) \end{pmatrix} = \mathbb{H}$

Thus the transition matrix must obey the differential equation

$$\frac{\partial \Phi(s,t)}{\partial s} = \mathbb{H}(s)\, \Phi(s,t) \qquad \ldots\ldots\ldots\ldots\ldots(2.8)$$

we now partition the matrix $\Phi(s,t)$ into the following matrix

$$\Phi(s,t) = \begin{pmatrix} \Phi_{11}(s,t) & \Phi_{12}(s,t) \\ \Phi_{21}(s,t) & \Phi_{22}(s,t) \end{pmatrix} \ldots\ldots\ldots\ldots\ldots(2.9)$$

using this fact and because the solution must also exist at $s = T$ we can form the solutions

$$\begin{aligned} x(T) &= \Phi_{11}(T,t)\, x(t) + \Phi_{12}(T,t)\, \lambda(t) + \hat{\Omega}(T,t) \\ \lambda(T) &= \Phi_{21}(T,t)\, x(t) + \Phi_{22}(T,t)\, \lambda(t) + \check{\Omega}(T,t) \end{aligned} \ldots\ldots(2.10)$$

where

$$\begin{pmatrix} \hat{\Omega}(T,t) \\ \check{\Omega}(T,t) \end{pmatrix} = \int_t^T \Phi(\xi,t) \begin{pmatrix} w(\xi) \\ w(\xi) \end{pmatrix} d\xi \qquad \ldots\ldots\ldots\ldots\ldots(2.11)$$

using this solution we can rewrite the boundary conditions as

$$(A\Phi_{12}(T,t) \;+\; B\Phi_{22}(T,t))\lambda(t) \;=\; d - A\hat{\Omega}(T,t) = B\check{\Omega}(T,t)$$
$$- (A\Phi_{11}(T,t) + B\Phi_{21}(T,t))\; x(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.12)$$

Here again the control law will be given by

$$u(t) \;=\; R(t)\; G(t)^T \lambda(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.13)$$

where $\lambda(t)$ is given by the equation (2.12).

To solve the control problem we need to calculate the matries $\Phi(T,t)$, $\hat{\Omega}(T,t)$ and $\check{\Omega}(T,t)$. These matries can be calculated in the following manner.

$\Phi(T,t)$ can be calculated by solving the equation

$$\frac{\partial \Phi(s,t)}{\partial s} \;=\; \Theta(s) \quad \Phi(s,t)$$

or rewriting

$$\frac{\partial(\Phi(s,t))_{ij}}{\partial s} \;=\; \sum_{\ell=1}^{2n}(\Theta(s))_{il}\;(\Phi(s,t))_{lj} \quad \ldots\ldots\ldots\ldots(2.14)$$

where $t \leqq s \leqq T$ and
$$i \;=\; (I,2,\ldots\ldots\ldots\ldots,2n)$$
$$j \;=\; (I,2,\ldots\ldots\ldots\ldots,2n)$$

subject to the initial equations

$$(\Phi(t,t))_{ij} \;=\; \delta_{ij} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.15)$$

Further the equations for terms $\hat{\Omega}(s,t)$ and $\check{\Omega}(s,t)$ can be formed as follows, let $\Psi(s,t) \;=\; \begin{pmatrix} \hat{\Omega}(s,t) \\ \check{\Omega}(s,t) \end{pmatrix}$

Thus from equation (2.11) we form the 2n integral equations

$$\int_t^s \Phi(s,\xi) \begin{pmatrix} \hat{w}(\xi) \\ \check{w}(\xi) \end{pmatrix} d\xi \;=\; \Psi(s,t) = \begin{pmatrix} \hat{\Omega}(s,t) \\ \check{\Omega}(s,t) \end{pmatrix} \ldots\ldots\ldots(2.16)$$

We can reduce this to a system of differential equations of the form

$$\frac{\partial \Psi(s,t)}{\partial s} = \Theta(s)\ \Psi(s,t) + \begin{pmatrix} \hat{w}(s) \\ \check{w}(s) \end{pmatrix} \quad \ldots\ldots\ldots\ldots(2.17)$$

for $t \leqq s \leqq T$

Once we have generated $\Psi(s,t)$ and $\Phi(s,t)$ for $t \leqq s \leqq T$ it is possible to obtain more than a solution to just a single boundary value problem. It may well be that one wishes to solve the problem for the case when $T \to \infty$. It is then of interest to solve it for a fixed value t and observe how the coefficients in the control law behave when $T \to \infty$. This is known as the floating time to go problem.

In a different approach the termination time T may be fixed and the control law may be required for successive value of t as $t \to T$. In this case one requires $\Phi(T,t)$ and $\Psi(T,t)$ on some interval $t_o \leqq t \leqq T$. To obtain this one generates $\hat{\Phi}(t,T)$ by solving the equations

$$\frac{\partial(\hat{\Phi}(t,T))_{ij}}{\partial t} = \sum_{l=I}^{2n} (\Theta(t))_{il}\ (\hat{\Phi}(t,T))_{lj} \quad \ldots\ldots\ldots\ldots(2.18)$$

where $i = (I,2,\ldots\ldots\ldots,2n)$
$j = (I,2,\ldots\ldots\ldots,2n)$

subject to the initial condition $\hat{\Phi}(T,T)_{ij} = \delta_{ij}$

In effect we are solving the problem by what is called the shrinking time to go. Thus we can express x(t) and $\lambda(t)$ as

$$\begin{pmatrix} x(t) \\ \lambda(t) \end{pmatrix} = \hat{\Phi}(t,T) \begin{pmatrix} x(T) \\ \lambda(T) \end{pmatrix} + \int_T^t \hat{\Phi}(t,\xi) \begin{pmatrix} w(\xi) \\ w(\xi) \end{pmatrix} d\xi \quad \ldots\ldots\ldots(2.19)$$

and

$$\begin{pmatrix} x(T) \\ \lambda(T) \end{pmatrix} = \hat{\Phi}^{-1}(t,T) \begin{pmatrix} x(t) \\ \lambda(t) \end{pmatrix} - \hat{\Phi}^{-1}(t,T) \int_T^t \hat{\Phi}(t,\xi) \begin{pmatrix} w(\xi) \\ w(\xi) \end{pmatrix} d\xi$$

SECTION II

PART III SOLUTION OF THE SHRINKING TIME TO GO PROBLEM

There exists a somewhat more elegant method for solving the shrinking time to go problem for linear time varying systems. The method which is to follow was first recognized by Kalman, who used Caratheodory fundamental equations and Hamilton Jacobi theory in its development. The same results are developed for the problem posed by Kalman using different techniques, calculus of variations, and then these results are extended to solve the more general non-homogeneous problem. This is done by an appropriate transformation.

In the problem treated it has been shown that the optimal control law is linear and is given by the equation (3.1)

$$u(s) = C(s,T)x(s) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.1)$$

The method presented formulates a way of solving for the coefficient $C(s,T)$ directly by use of a nonlinear differential equation of the Riccati type treated as an initial value problem. Further it is required in Kalman's problem that natural boundary conditions exist at $s = T$. In situations in which it is applicable, this method yields an excellent way of solving control problems. It is also very interesting to note, and which will be shown later, that the simple regulator problem solved by this method is the dual to the linear time varying filter problem.

The problem to be solved assumed initially that the desired equilibrium state of the system is zero, we then generalize the problem to the case where the system is to follow as closely as possible some desired output.

The performance criterion is given by

$$V(s,T,x(s),u(s)) = \mathcal{V}(x(T),T) + \int_t^T L(x(s),u(s),s)ds \quad \ldots.(3.2)$$

such that at $s = T$, "V" is given by

$$V(T,T,x(T),U(T)) = \mathcal{V}(x(T),T) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3.3)$$

The following assumption on the regulator problem for desired output of zero are;

(a) The plant is governed by the linear differential equations

$$\frac{dx(s)}{ds} = F(s)\ x(s) + G(s)\ u(s) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3.4)$$

(b) The loss functions are quadratic forms in the state and control variables, and given by the equations (3.5)

$$L(x(s),u(s),s) = I/2\left[\|H(s)x(s)\|^2_{Q(s)} + \|u(s)\|^2_{R(s)}\right] \quad \ldots\ldots(3.5)$$

where Q(s), R(s) are symmetric positive definite matries of class C^2 in "s".

Investigating the Euler equations

$$\begin{aligned}\frac{dx(s)}{ds} &= F(s)\ x(s) + G(s)\ R^{-1}(s)\ G^T(s)\lambda(s) \\ \frac{d\lambda(s)}{ds} &= H^T(s)\ Q(s)\ H(s)\ x(s) - F^T(s)\lambda(s)\end{aligned} \quad \ldots\ldots\ldots\ldots\ldots.(3.6)$$

and making the change of variables $\xi(s) = -\lambda(s)$ the equations become

$$\begin{aligned}\frac{dx(s)}{ds} &= F(s)\ x(s) - G(s)R^{-1}(s)G^T(s)\xi(s) \\ -\frac{d\xi(s)}{ds} &= H^T(s)Q(s)H(s)x(s) + F^T(s)\xi(s)\end{aligned} \quad \ldots\ldots\ldots\ldots\ldots.(3.7)$$

And the control law becomes from equation (2.13)

$$u(s) = -R^{-1}(s)G^T(s)\xi(s) \quad \ldots\ldots(3.8)$$

Because it is desired to form a control law of equation ( ) a relationship between x(s) and $\xi(s)$ must be established. Recognizing that the Euler equations suggest that such a relationship exists we let.

$$\xi(s) = P(s,T)\,x(s) \quad \ldots\ldots(3.9)$$

differentiating this relationship with respect to "s" we get equation (3.10).

$$\frac{d\xi(s)}{ds} = \frac{dP(s,T)}{ds}x(s) + P(s,T)\frac{dx(s)}{ds} \quad \ldots\ldots(3.10)$$

upon substitution for $\frac{d\xi(s)}{ds}$ and $\frac{dx(s)}{ds}$ for equations (3.7) we get

$$-H^T(s)Q(s)H(s)\,x(s) - F^T(s)\xi(s) = \frac{dP(s,T)}{ds}\,x(s) + P(s,T)F(s)\,x(s)$$

$$-P(s,T)G(s)R\ (s)G^T(s)\xi(s) \quad \ldots\ldots(3.11)$$

which becomes the well-known Riccati equation stated by Kalman

$$-\frac{dP(s,T)}{ds} = P(s,T)F(s) + F^T(s)P(s,T) + H^T(s)Q(s)H(s)$$
$$- P(s,t)G(s)R^{-1}(s)G^T(s)P(s,T) \quad \ldots\ldots(3.12)$$

where P(s,T) is a symmetric matrix to avoid trivia.

This is the same result as that obtained by Kalman using different techniques.

We now generalize the above problem to treat the case where the desired output is given by z(s) and the loss function is given by

$$L(x(s),u(s)) = I/2\left[\left\| H(s)x(s) - z(s)\right\|^2_{Q(s)} + \left\|u(s)\right\|^2_{R(s)}\right] \ldots(3.13)$$

The plant is assumed to be given by

$$\frac{dx(s)}{ds} = F(s)\ x(s) + G(s)u(s) + K(s)\ w(s) \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.14)$$

The resulting Euler equations are given by

$$\begin{pmatrix} \frac{dx(s)}{ds} \\ -\frac{d\xi(s)}{ds} \end{pmatrix} = \begin{pmatrix} F(s) & -G(s)R^{-1}(s)G^T(s) \\ H^T(s)Q(s)H(s) & F^T(s) \end{pmatrix} \begin{pmatrix} x(s) \\ \xi(s) \end{pmatrix} + \begin{pmatrix} \hat{w}(s) \\ \check{w}(s) \end{pmatrix} \ldots\ldots(3.15)$$

where $\hat{w}(s) = K(s)\ w(s)$ and $\check{w}(s) = H^T(s)Q(s)\ z(s)$

If we can put this in the form of the equations (3.7)

$$\begin{pmatrix} \frac{dx^*(s)}{ds} \\ -\frac{d\xi^*(s)}{ds} \end{pmatrix} \begin{pmatrix} F(s)^* & -R^*(s) \\ Q^*(s) & F^{T*}(s) \end{pmatrix} \begin{pmatrix} x^*(s) \\ \xi^*(s) \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.16)$$

then the Riccati equation follows immediately as above.

The above form can be obtained under the transformations

Let $\quad x^*(s) = \begin{matrix} x(s) \\ I \end{matrix}$ and $\xi^*(s) = \begin{pmatrix} \xi(s) \\ \xi_0(s) \end{pmatrix}$

$F^*(s) = \begin{vmatrix} F(s) & \hat{w}(s) \\ 0 & 0 \end{vmatrix} \qquad R^*(s) = \begin{vmatrix} G(s)R\ (s)G^T(s) & 0 \\ 0 & 0 \end{vmatrix}$

$Q^*(s) = \begin{vmatrix} H^T(s)Q(s)H(s) & \check{w}(s) \\ \check{w}(s) & 0 \end{vmatrix} \qquad G^*(s) = \begin{vmatrix} G(s) \\ 0 \end{vmatrix}$

The result Riccati equation is given by

$$-\frac{dP^*(s,T)}{ds} = P^*(s,T)\ F^*(s) + F^{*T}(s)P^*(s,T) + Q^*(s) - P^*(s,T)R^*(s)P^*(s,T) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.17)$$

Kalman has suggested that the performance criterion can be expressed in the nonnegative definite form.

$$V(x,u,s) = \frac{1}{2} \, ||x(s)||_{P(S,T)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.18)$$

if $\mathcal{V}(x(T)) = \frac{1}{2} \, ||x||^2_A$ where A is symmetric, nonnegative definite

This is a very convenient form, because (as we will show shortly) $V(x,u,s)$ is then a Lyapunov function.

To show that the expression (3.18) obeys the necessary conditions, recall that by definition $\xi(s) = P(s,T)\, x(s)$. We then differentiate $V(x,u,s)$ with respect to "s" and show that the right-hand side is just the Hamiltonian as discussed in Section I Part II. If this relationship holds then the expression (3.18) is the performance criterion.

$$V(x,u,s) = \frac{1}{2} x^T P(s,T)\, x = \frac{1}{2} x^T \xi(s)$$

$$V_s(x,u,s) = \frac{1}{2} \frac{dx^T}{ds} \xi(s) + \frac{1}{2} x^T \frac{d\,\xi(s)}{ds}$$

Substituting the value for $\frac{d\xi}{ds}$ and $\frac{dx}{ds}$ from (3.7) we get

$$V_s(x,u,s) = -\mathcal{H}^* = -L(x,u,s) - \langle \dot{x} , \xi(s) \rangle$$

Furthermore, if a plant is completely controllable then the limit as $T \longrightarrow \infty$ exists. We denote this limit by $\overline{P}(t)$ and denote the solution of the Riccati solution (3.12) by $\Pi(s;A;T)$. Therefore

$$\lim_{T \to \infty} \Pi(s;A,T) = \overline{P}(s) \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.19)$$

Furthermore, if this limit exists the performance criterion becomes

$$V(t;x,u,\infty) = \lim_{T \to \infty} V(t_o;x,u,T) = \|x\|^2_{\overline{P}(t_o)} \quad \ldots\ldots\ldots\ldots(3.20)$$

and the control law becomes

$$u(s) = -R^{-1}(s)G^T(s)\overline{P}(s)x(s) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.21)$$

In this part we consider the stability of the shrinking time to go problem as stated by Kalman. It will be shown that if a plant is uniformly completely controllable and uniformly completely observable, plus the additional conditions stated in the following theorem, the plant is uniformly asymptotically stable. If we weaken the conditions to completely controllable and completely observable, the plant is (non-uniformly) asymptotically stable.

Stability Theorem; Consider a plant with control law (3.8) which is uniformly completely controllable and uniformly completely observable. In addition assume,

(i) $L(x,u,s)$ is quadratic and of form (3.5)

(ii) $\alpha_6 I \geq Q(s) \geq \alpha_4 I > 0$

(iii) $\alpha_7 I \geq R(s) \geq \alpha_5 I > 0$

(iv) $\|\Phi(s,T)\| \leq (\alpha_3 |s-T|)$ for all s, T

Then: the controlled plant is uniformly asymptotically stable and $V(x,s,\infty)$ is one of its Lyapunov functions.

Proof: From Lyapunov's Theorem (see Appendix B) if $V(x,s,\infty)$ is (a) bounded from above and (b) bounded from below by increasing functions of $\|x\|$. (c) the derivative $\dot{V}$ of V along optimal motion of the plant is negative definite and (d) $V \longrightarrow \infty$ as $\|x\| \longrightarrow \infty$ then $V(x,s,\infty)$ is a Lyapunov function. Then by Theorem (B-2) the plant is uniformly asymptotically stable.

<u>Part A</u> Recalling the proof for complete controllability that the control function $u^1(s)$ could be properly defined by

$$u^1(s) = -G^T(s)\Phi^T(t_o,s)\, W^{-1}(t_oT)\, x_o \quad \ldots\ldots\ldots\ldots\ldots(3.22)$$

if we substitute this into the equation of motion (B-10)

$$\phi_u^1(s;x_o,t_o) = \Phi(s,t_0)x_o - \int_{t_o}^{s}\Phi(s,\sigma)\, G(\sigma)G^T(\sigma)\Phi^T(t_o,\sigma)W^{-1}\, x_o \ldots\ldots\ldots\ldots(3.23)$$

Now using the fact that $\Phi(s,\sigma) = \Phi(s,t_o)\Phi(t_o,\sigma)$ the equation (3.23) becomes

$$= \Phi(s,t_o)x_o - \Phi(s,t_o)W(t_o,s)W^{-1}(t_o,T)x_o$$

$$= \Phi(s,t_o)\left[I - W(t_o,s)W^{-1}(t_o,T)\right]x_o \ldots\ldots\ldots\ldots\ldots\ldots(3.24)$$

because $T \geq s$ we assume that $\|I - W(t_o,s)W^{-1}(t_o,T)\| \leq I$

Define $z(s) = \Phi(T,t_o)\left[I - W(t_o,s)W^{-1}(t_o,T)\right]x_o \ldots\ldots\ldots(3.25)$

$$\|z(s)\| \leq \|\Phi(T,t_o)\| \cdot \|I - W(t_o,s)W^{-1}(t_o,T)\| \cdot \|x_o\|$$

Therefore

$$\|z(s)\| \leq \alpha_8(\sigma)\, \|x_o\| \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.26)$$

Because the region defined is given by $[t_o, t_o + \sigma]$ (see uniform completed controllability) where σ is a positive constant which transfers x_o to zero at or before $T = t_o + \sigma$. Hence $\alpha_8(\sigma)$ will be a constant.

NOTE: (t_o = t of the general problem)

We now investigate $V(x,t_o,\infty)$. Because $V(x,t_o,\infty) \leq$ $V(x,t_oT)$ we have that

$$V(x,t_o\,\infty) \leq \int_{t_o}^{T} L(x,u,s)ds + \frac{1}{2} \| x(T)\|^2_A \quad \text{.............(3.27)}$$

where A is a symmetric negative matrix. Using this along with (i), (ii) and (iii) we get

$$V(x,t_o\,\infty) \leq \int_{t_o}^{T}\left[\alpha_6 \|H(s)\Phi(s,T)z(s)\|^2 + \alpha_7\|u^1(s)\|^2\right] ds$$

$$\leq \int_{t_o}^{T} \left[\alpha_6\alpha_8 \|H(s)\Phi(s,T)\|^2 \cdot \|x_o\|^2 + \alpha_7\|u^1(s)\|^2\right] ds$$

$$\leq \int_{t_o}^{T} \Big[\alpha_6\alpha_8 \Phi^T(s,T)H^T(s)H(s)\Phi(s,T)\|x_o\|^2$$

$$+ \alpha_7(x_o^T W^{-1}(to,T)^T \Phi(t_o,s)G(s)G^T(s)\Phi^T(t_o,s) W^{-1}x_o\Big] ds$$

$$\leq \alpha_6\alpha_8 M(T,t_o)\|x_o\|^2 + \alpha_7 x_o^T W^{-1}({}^Tt_o,T)\, x_o$$

$$\leq \alpha_6\alpha_8 M(T,t_o)\|x_o\|^2 + \alpha_7 \|x_o\|^2 W^{-1}(t_o,T) \quad \text{..............(3.28)}$$

Thus by uniform complete controllability, and uniform complete observability we have that

$$V(x,t_o,\infty) \leq \left[\alpha_6\alpha_8\alpha_1^*(\sigma) + \alpha_7\alpha_o(\sigma)\right]\|x_o\|^2$$

$$\leq \alpha_9 \|x_o\|^2 \quad \text{...................(3.29)}$$

which proves part (a)

<u>Part B</u> We must show that $V(x,t_o\infty)$ is bounded from below for increasing $\|x\|$

From $V(x,t_o\infty) = \|x\|^2_{\bar{p}(t_o)}$

Define $V(x,t_o\infty) = \alpha_{10}(t_o)\|x\|^2$ ....(3.30)

if $V(x,t_o,\infty)$ is bounded from below, it must be the case that

$\alpha_{10}(t_o) \geqq \alpha_{11} \geqq 0$

Assume that this is not true and that in fact there is some $\epsilon(x,s)$ defined by

$$\|x\|^2\,\epsilon(x,s) = \int_{t_o}^{T} \|u\|^2 ds \leqq \alpha_5^{-1} \int_{t_o}^{T} \|u\|^2_{R(s)}\, ds$$
$$\leqq V(x,t_o\infty) \quad \ldots\ldots\ldots\ldots\ldots(3.31)$$

such that $\epsilon(x,s)$ can be made as small as desired by a proper choice of $x\ t_o$.

Define

$$z(s) = \int_{t_o}^{T} \Phi(t_o,s)G(s)u(s)\, ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.32)$$

and by Schwary inequality

$$\|z(s)\|^2 \leqq \left(\int_{t_o}^{T} \|\Phi(t_o,s)G(s)\|^2 ds\right)\left(\int_{t_o}^{T} \|u(s)\|^2 ds\right) \ldots\ldots(3.33)$$

by uniform complete controllability

$$\|z(s)\|^2 \leqq W(t_o,T) \int_{t_o}^{T} \|u(s)\|^2 ds$$

$$\leqq \alpha_1(\sigma)\|x_o\|^2 \,\epsilon(x_o,t) = \alpha_{12}(\ \ )\|x_o\|^2 \ldots\ldots\ldots\ldots\ldots(3.34)$$

Utilizing this estimate, we find with the addition of (i,) (ii) and (iii)

$$V(x,t_o,\infty) \geqq \int_{t_o}^{T} \alpha_4 \left\| H(s)\Phi(s,t_o)\left[x_o + \ \ (s)\right]\right\|^2 ds$$

$$\geqq \int_{t_o}^{T} \alpha_4 \Big\{ \|H(s)\Phi(s,t_o)\|^2 \|x_o\| - \|H(s)\Phi(s,t_o)\|^2 \cdot \|z(s)\|^2 \Big\}\, ds$$

$$\geqq \alpha_4 \|\Phi(T,t_o)x_o\|^2 M(T,t_o) - \alpha_{12}(\ \ )M(T,t_o)\|x_o\|^2$$

$$\geqq \alpha_4 \left[\alpha_3^{-2}(\sigma)\alpha_o^*(\sigma) - \alpha_1^*(\sigma)\,\alpha_{12}(\ \ )\ \ \|x_o\|^2\right.$$

$$\geqq \left[\alpha_{13} - \alpha_{14}\right] \|x_o\|^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.25)$$

which contradicts the assumption that α_{12} hence ϵ can be made arbitrarily small.

Part C Since G and H are allowed to be singular, we cannot prove that V is negative definite. However, from Part B we have that

$$V(Q_u(T,x,t_o),\ T,\ \infty) - V(x,t_o\ \infty) \leqq -\left[\ \alpha_{13}\ - \alpha_{14}\right] \|x\|^2 \quad \ldots(3.36)$$

we also have that

$$V(Q_u(T,x,t_o),\ T,\ \infty) - V(x,t_o,\infty) \leqq -\alpha_5 \|x\|^2$$

$$\text{setting } \alpha_{15} = \alpha_5 \alpha_{13} / (\alpha_5 + \alpha_{14}) > 0 \quad \ldots\ldots\ldots(3.37)$$

we have

$$V(Q_u(t_o, + \sigma, \cdot x,t_o),\ t_o + \sigma,\infty) - \overset{\circ}{V}(x,t_o,\infty)$$

$$\leqq -\alpha_{15}\|x\|^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3.38)$$

which shows that V is decreasing along any interval of time of length σ unless $x_o = 0$

Part D This is true in view of $V \geqq \alpha_{12}\|x_o\|^2$

Q.E.D.

SOLUTION OF THE RICCATI EQUATION

The Riccati Equation

$$\frac{-dP(s,T)}{ds} = P(s,T)\,F(s) + F^T(s)P(s,T) + H^T(s)Q(s)H(s) - P(s,T)G(s)R^{-1}(s)G^T(s)P(s,T)$$

can be solved as an initial value problem directly by the solution of the 2n canonical Euler equations as illustrated before.

For the Euler equations

$$\begin{pmatrix} \frac{dx(s)}{ds} \\ \frac{-d\xi(s)}{ds} \end{pmatrix} = \begin{pmatrix} F(s) & -G(s)R^{-1}(s)G^T(s) \\ H^T(s)\,Q(s)H(s) & + F^T(s) \end{pmatrix} \begin{pmatrix} x(s) \\ \xi(s) \end{pmatrix}$$

Let the transition matrix for the 2n Euler equations be $\Gamma(s,T)$, we then partition this matrix into

$$\begin{pmatrix} \gamma_{11}(s,T) & \gamma_{12}(s,T) \\ \gamma_{21}(s,T) & \gamma_{22}(s,T) \end{pmatrix}$$

It is known that $\xi(s)$ and $x(s)$ are related by the following equation - $\xi(s) = P(s,T)x(s)$

and from the above it must be the case that

$$\frac{d\Gamma(s,T)}{ds} = \begin{pmatrix} F(s) & -G(s)R^{-1}(s)G^T(s) \\ H^T(s)Q(s)H(s) & F^T(s) \end{pmatrix} \Gamma(s,T)$$

so the solution

$$\begin{pmatrix} X(s) \\ Z(s) \end{pmatrix} = \Gamma(S,T) \begin{pmatrix} X(T) \\ Z(T) \end{pmatrix}$$

$$X(s) = \gamma_{11}(s,T)\ X(T) + \gamma_{12}(s,T)\ Z(T)$$

$$Z(s) = \gamma_{21}(s,T)\ X(T) + \gamma_{22}(s,T)\ Z(T)$$

because $Z(T) = P(T,T)\ X(T)$

we have

$$X(s) = \left[\gamma_{11}(s,T) + \gamma_{12}P(T,T)\right] X(T)$$

$$Z(s) = \left[\gamma_{21}(s,T) + \gamma_{22}P(T,T)\right] X(T)$$

Hence $P(s,T) = Z(s)\ X^{-1}(s)$

Therefore

$$P(s,T) = \left[\gamma_{21}(s,T) + \gamma_{22}(S,T)P(T,T)\right]\left[\gamma_{11}(S,T) + \gamma_{12}P(T,T)\right]^{-1}$$

For proof of these results see [14].

EXAMPLE PROBLEM II - 1

To illustrate the developed theory for linear systems with linear control laws consider the following problem.

The plant is given by the equations

$$\frac{dx}{ds} = f_{11}x + u$$

$$y = x$$

NOTE: This is the first order case and in general $x = x_1$; $y = y_1$, and $u = u_1$; however for convenience we drop the subscript.

We ~~The~~ wish to optimize this plant subject to the performance criterion

$$V = \int_t^T \left[q_{11}x^2 + r_{11}u^2 \right] ds$$

Letting $q_{11} = q$; $r_{11} = r$; and $f_{11} = f$, the resulting Euler equations are

$$\frac{dx}{ds} = fx + \frac{1}{r}\lambda$$

$$\frac{d\lambda}{ds} = qx - f\lambda$$

Rewriting in matrix form

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} f & \frac{1}{r} \\ q & -f \end{pmatrix} \cdot \begin{pmatrix} x \\ \lambda \end{pmatrix}$$

From the matrix $\Theta(S,t)$

$$= \begin{pmatrix} \cosh a(s-t) + \frac{f}{a} \sinh a(s-t), & \frac{1}{ra} \sinh a(s-t) \\ \frac{q}{a} \sinh a(s-t), & \cosh a(s-t) - \frac{f}{a} \sinh a(s-t) \end{pmatrix}$$

where $a^2 = f^2 + \frac{q}{r}$

The control law for this problem is given by

$$u(t) = \frac{1}{r} \cdot \lambda(t)$$

or using Kalman's solution

$$u(t) = -\frac{1}{r} \quad P(t) \quad x(t)$$

Case I Natural Boundary Conditions

Natural boundary conditions are given by

$$\lambda(T) = P(T) = 0$$

Floating Time to go Solution

Using the control law $u(t) = \frac{1}{r} \lambda(t)$, it is necessary to find $\lambda(t)$ in terms of $x(t)$. From the transition matrix $\lambda(s)$ is given by

$$\lambda(s) = \frac{q}{a} \sinh a\,(s-t)x(t) + \cosh a(s-t)\lambda(t)$$
$$- \frac{f}{a} \sinh a\,(s-t)\lambda(t)$$

Applying the boundary condition $\lambda(T) = 0$ gives a relation between $\lambda(t)$ and $x(t)$

$$\lambda(t) = \frac{\frac{q}{a} \sinh\ a(T-t)}{\frac{f}{a} \sinh\ a(T-t) - \cosh a(T-t)} x(t)$$

In this case we consider fixed values of t and successive values of T to observe the limit of C(T,t) as $T \rightarrow \infty$, where C(T,t) is given by

$$q \frac{e^{a(T-t)} - e^{-a(T-t)}}{e^{a(T-t)}(\frac{f}{a} - a) - e^{-a(T-t)}\ (f + a)}$$

In the limit as $T \rightarrow \infty$, $C(T,t)$ tends to $\frac{q}{f-a}$

Shrinking Time to go Solution

Because we have subjected this problem to natural boundary conditions, Kalman's solution may be used. From which it follows that

$$P(t,T) = \frac{\left[\gamma_{21}(t,T) + \gamma_{22}(t,T)P(T,T)\right]}{\left[\gamma_{11}(t,T) + \gamma_{12}(t,T)P(T,T)\right]}$$

Because $\lambda(T) = 0$ and $x(T) \neq 0$ it must be the case that $P(T,T) = 0$.

In the solution of the shrinking time to go problem, T is fixed and successive values of t are considered for $t \leq T$.

$$P(t,T) = \frac{q\left(e^{a(t-T)} - e^{-a(t-T)}\right)}{(a+f)\, e^{a(t-T)} + (a-f)e^{-a(t-T)}}$$

Taking the limit $P(t,T)$ tends to $\frac{g}{f-a}$

For the steady state case, an easier method is available for finding the limit of the shrinking time to go problem. This solution is obtained by setting the right-hand side of the Riccati equation equal to zero and solving the resulting quadratic equation for the nonnegative definite solution. This is given by,

$$P = \frac{q}{a-f}$$

Case II Boundary Conditions $x(T) = 0$

Floating Time to go Solution

From the transition matrix $x(s)$ is given by $x(s) =$

$$\left[\cosh a(s-t) + \frac{f}{a} \sinh a(s-t)\right] x(t)$$

$$+ \frac{1}{ra} \sinh a(s-t)\, \lambda(t)$$

Subjecting this equation to the boundary condition $x(T) = 0$ gives the desired relationship between $\lambda(t)$ and $x(t)$.

This relationship is given by

$$\lambda(t) = \left[-ra \cosh a(T-t) + \frac{f}{a} \sinh a(T-t)\right] x(t) \Big/ \sinh a(T-t)$$

In this case, as the time to go becomes infinitesimal, there remains some error, hence the control effort asked for becomes unbounded in an attempt to satisfy the end boundary conditions. This can be seen by evaluating $C(T,t)$, at $t = T$ where the $\sinh a(T-t)$ becomes zero. Hence, $C(T)$ becomes infinite. In this case, the control law is said to have finite escape time.

Similarly we could solve this problem by using the shrinking time to go method. Note however, that because $P(T)$ is not defined. Kalman's solution is not applicable.

SECTION II

PART IV CONSTRAINTS IN VARIATIONAL PROBLEMS AND SOME NONLINEAR CONTROLLERS

In the procedures developed thus far no account has been made for possible limitations on the allowable range of the state of control variables, their derivatives or integrals. The treatment of a large variety of constraints can be done by the use of Lagrange multipliers, this method has already been employed to convert a performance criterion constrainted by the plant into an unconstrainted performance criterion. Constraints on the plant and the control functions which are most easily solved by the use of Lagrange multipliers are generally of the type

$$q(t;x(t),u(t)) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.1)$$

where "q" is an r-dimensional vector function.

The performance criterion can then be modified by the addition of the term

$$\int_t^T \rho(s) \;\; q(s;x(s),u(s)) \; ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.2)$$

(where $\rho(s)$ is the Lagrange multiplier).

and require that the performance criterion be stationary with respect to $\rho(t)$ as well as the other variables. The modified performance criterion will be stationary only if equation (4.2) is satisfied; then the value of the modified criterion will be equal to the original "V". This procedure will introduce into the Euler Lagrange equation an extra set of algebraic equations, equation (4.1) in the variable $\rho(t)$.

Another possible form of a constraint is the inequality integral

$$\int_t^T q_i(s;x(s),u(s))ds \leq \Gamma_i \quad \text{.........................(4.3)}$$

where $(i = 1,2,3, \ldots\ldots\ldots\ldots, n)$

be satisfied. One again can introduce a set of constant Lagrange multipliers "μ" and add to the performance criterion a term

$$\int_t^T \mu^T q(s;x(s),u(s))\, ds \quad \text{.............................(4.4)}$$

and carry out the optimization for a range of values of the multipliers "μ" expressing the optimum "V" as a function of "μ". One then performs what is but static optimization with respect to "μ" subject to the constraints of equation (4.3).

Another inequality constraint which is the most important type, from a physical point of view, is the form

$$u_i^l \leq u_i(t) \leq u_i^u \quad (i = 1,2,3,\ldots\ldots\ldots,n) \quad \text{........(4.5)}$$

$$x_i^l \leq x_i(t) \leq x_i^u$$

where the superscripts u and l designate upper and lower limits or bounds respectively. These types of constraints, especially the condition on $u_i(t)$ appear in physical situations. Some examples of problems of this type that have been solved to date are the minimum time, fuel and energy problems. There are several ways of incorporating these inequalities. The most simple, although not the most accurate, is to substract from "V" a term of the form

$$\int_t^T P_{u_i}(u_i(s))ds \quad \text{and} \quad \int_t^T P_{x_i}(x_i(s))ds \quad \text{...........(4.6)}$$

where P_{u_i} and P_{x_i} are penalty functions for the control and state functions respectively. These penalty functions are such that they provide a large penalty function to "V" if their arguments are about to violate the bounds expressed by equation (4.5) but little if their arguments are within bounds. The most convenient function of this kind, that applies to a large variety of practical problems, has been found to be

$$P_{u_i}(u_i) = \begin{cases} k_2u^2 + (k_2 - k_I)(I + 2u) & u > -I \\ k_Iu^2 & |u| < I \\ k_2u + (k_2 - k_I)(I - 2u) & u > I \end{cases} \qquad \text{.............(4.7)}$$

or

$$P_{u_i}(u_i) = (u)^r$$

where

$$u = \frac{u_i - I/2(u_i^u - u_i^l)}{u_i^u - u_i^l}$$

and $k_2 > k_I$ and r is as high an even power as is necessary to achieve a sharp enough constraint.

As was pointed out this is just an approximate method. However, it has the advantage that it is easy to apply in complicated problems and can be used to constrain both the state and the control variables.

Another way of handling inequality constraints of the nature described above which leads to an exact solution is to use a change of variables. If y(t) is some variable which is to be constrainted between two limits y^l and y^u, one introduces a new variable z(t) such that

$$y(t) = q(t;z(t)) \qquad \text{.............................(4.8)}$$

and $q(t;z(t))$ remains in the bounded region specified by the constraints. Substituting $q(t;z(t))$ for $y(t)$ one then performs the optimization with to $z(t)$, thus for any $z(t)$ that may result, $y(t)$ will remain within the constrainted limits.

To illustrate the use of the method presented for the solution of problems subject to constraints, let us consider the problems of minimum time, fuel and energy. We shall solve these problems for necessary conditions by the use of Euler equations.

The Minimum Time Problem

The minimum time problem, or the "bang-bang" problem has been examined in detail during the past ten years and the solution of an idealized relay is well-known. The reason that this problem is presented here is to compare the developed theory with a well established result.

The plant described by equations (4.9-10)

$$\frac{dx(s)}{ds} = Fx(s) + Gu(s) \qquad \text{........................(4.9)}$$

$$y(s) = Hx(s) \qquad \text{........................(4.10)}$$

and the control $u(s)$, restricted in magnitude by equation (4.11).

$$u_j(s) \leq c_j \qquad \text{........................(4.11)}$$

where $j = I,2, \ldots\ldots\ldots\ldots \; p$

for simplicity we will assume that all c_j are eqyal to "1".

It is assumed that the system is controllable (see Section I, Part 2). The initial and final states to the plant are given by x_o and x_f respectively. With no loss of generality assume $t = 0$.

For the minimum time problem the loss function will be given by $L(s;x(s),u(s)) = I$ since

$$\int_0^T L(s(s),u(s))ds = T$$

To account for the constraints imposed on the control function u(s) we add the penalty function of the form $\sum_{j=I}^{m} \left[u_j(s)\right]^r$ to the loss function where each component $u_j(s)$ of u(s) is penalized by the function $u_j(s)^r$. r is given to be some even power as high as necessary to achieve a sharp enough constraint. The new loss function will be given by $\left(1 + \sum_{j=I}^{m} \left[u_j(s)\right]^r\right)$.

The resulting Euler equation for this system are

$$\frac{dx(s)}{ds} = Fx(s) + Gu(s)$$

$$\frac{d\lambda(s)}{ds} = - F^T\lambda(s) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.12)$$

$$G^T\lambda(s) = ru(s)^{r-1}$$

Because u(s) must be subjected to the constraint $|u_j(s)| \leq 1$ we may rewrite the last equation as

$$u(s) = \text{sgn}\left(\underline{q}(s)\right)\left|\frac{I}{r}G^T\lambda(s)\right|_I^{\frac{1}{r-1}}$$

and in the limit as $r \rightarrow \infty$ the term $\left|\frac{x}{n}\right|^{\overline{n-1}}$ becomes equal to I

and a necessary condition for the minimum time problem is that

$$u(s) = \text{sgn}\left[\underline{q}(s)\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.13)$$

This situation is depicted in Figure (II-4.1,A)

NOTE: (q (s) is defined to be G^T (s))

The Minimum Fuel Problem

There are many problems where the control variables u(s) are directly proportional to the rate of flow of mass. For example, the control signal may be the flow from a gas jet for the attitude control of a space vehicle or the exhaust from a missile. Almost invariably in these problems, the total fuel available is limited; therefore, it is desirable to accomplish each control correction with a minimum amount of fuel.

In this case, the loss function will be given by

$Lx(s), u(s) = \sum_{j-1}^{m} \left| u_j(s) \right|$ To solve this problem, once again we add for each input a constrainted function $u_j(s) = \sin\theta_j$ resulting in new loss function namely $\sum_{j=1}^{m} |\sin\theta_j|$ from which the Euler equations are

$$\frac{dx(s)}{ds} = Fx(s) + Gu(s)$$

$$\frac{d\lambda(s)}{ds} = - F^T\lambda(s) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.14)$$

$$G^T \lambda(s) = L_u$$

where $\left| u_j(s) \right| \leq 1$

We now define the vector "q" to be equal to $G^T \lambda(s)$, the last equation of (4.14) becomes

$$\cos\theta \quad q(s) = \cos\theta \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4.15)$$

$$\text{for } \left| u_j(s) \right| \leq 1$$

Case I: If $\left| q_j(s) \right| \geq 1$

then $u_j(s) = \operatorname{sgn}\left[q_j(s) \right]$

Case II: if $\left| q_j(s) \right| < 1,$

then $u_j(s) = 0$

Thus, for minimum fuel operations, the control law is given by

$$u_j(s) = 0 \quad \text{if} \quad \left| q(s) \right| < 1$$

$$u_j(s) = \operatorname{sgn}\left[q_j(s) \right] \quad \left| q_j(s) \right| \geq 1 \quad \ldots\ldots\ldots\ldots (4.16)$$

From which we can conclude that the values of optimal control functions are $\{1, 0 \text{ or } -1\}$.

This situation is depicted in Figure (II 4.1,B) below.

The Minimum Energy Problem

There exists a class of problem s for which the square of the control signal is proportional to power, and the time integral of the square of the control signal is a measure of the energy dissipated. As an example suppose that the control u(s) is an electric signal deriving its energy from a battery, and it is desired to accomplish a given control action using a minimum amount of energy provided by the battery.

The loss function in this case will be a function of $u(s)^2$ and given by

$$L(x(s),u(s)) = 1/2 \sum_{j-1}^{m} \left[u_j(s) \right]^2 = \frac{1}{2} \langle u(s),u(s) \rangle \qquad \text{...........(4.17)}$$

subject to the constraint $\left| u_j(s) \right| \leq 1$

To account for the bounded condition imposed on u(s) let $u_j(s) = \sin\theta_j$ this condition remains in the bounded condition imposed.

The control law in this case will be given by

$$\cos\theta \quad q(s) = \cos\theta \quad \sin\theta$$

From this equation we see that several conditions exist

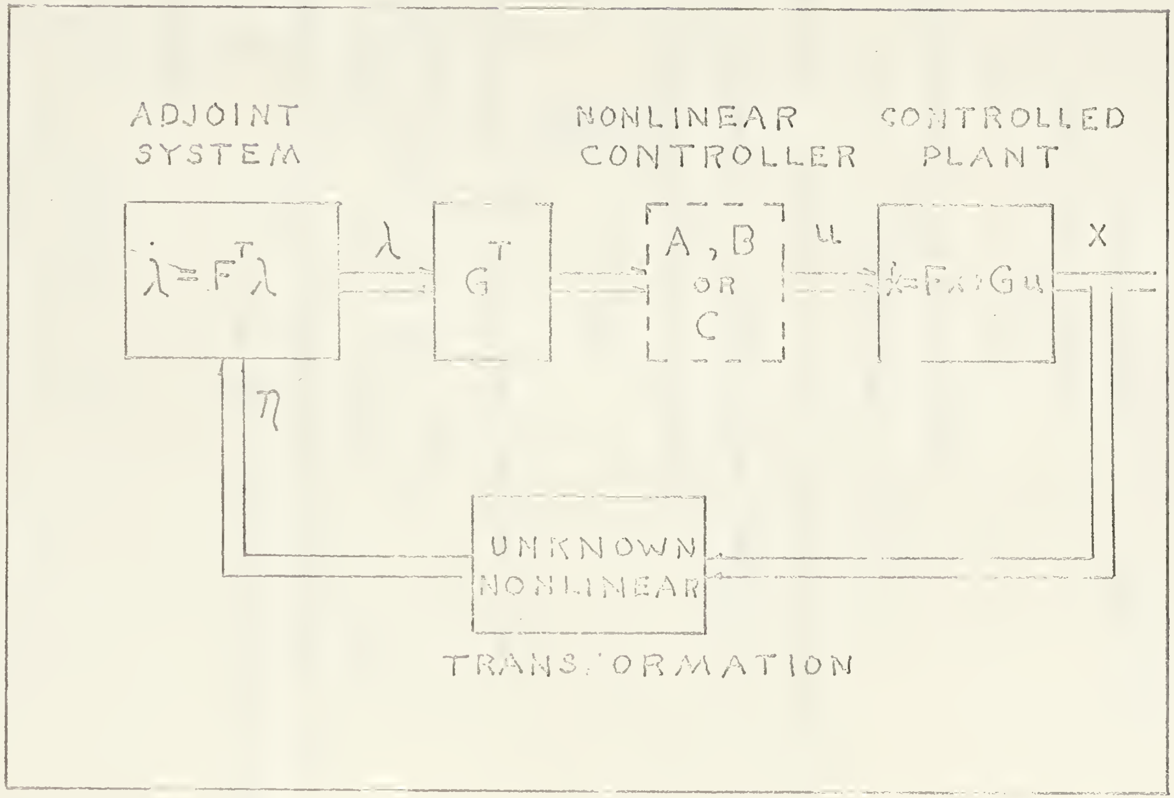
Case I: if $\left| q_j(s) \right| \geq$ I then it must be the case that $\cos\theta_j = 0$ hence $\sin\theta_j = \text{I}$, hence $u_j(s) = \text{sgn}\left\{ q_j(s) \right\}$

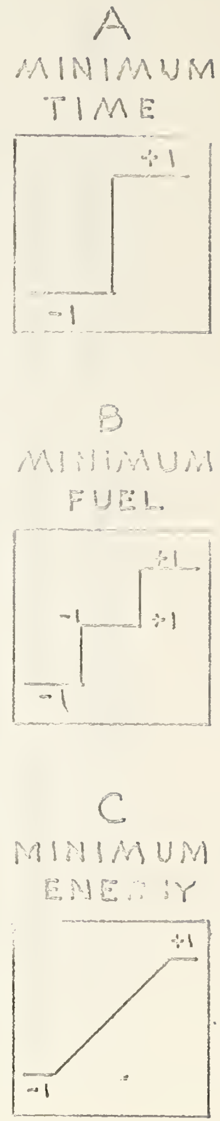
Case 2: if $\left| q_j(s) \right| <$ I then the equation exists for all values $j = 1, \ldots\ldots m$, hence $u_j(s) = q_j(s)$

In conclusion we see that the optimal control law for minimum energy is given by

$$\begin{aligned} u_j(s) &= q_j(s) && \text{if } \left| q_j(s) \right| < \text{I} \\ u_j(s) &= \text{sgn } q_j(s) && \text{if } \left| q_j(s) \right| \geq \text{I} \end{aligned} \qquad \text{............(4.18)}$$

NONLINEAR CONTROLLERS FOR MINIMUM TIME FUEL AND ENERGY





<u>EXAMPLE PROBLEM II - 4.1</u> A Complete Solution

The Minimum Fuel Control System for a second order system. We have, using Calculus of Variations, developed a necessary condition for the minimum fuel problem. The conditions are,

$$u_j(s) = 0 \quad \text{if} \quad |q_j(s)| < 1$$

$$u(s) = + \operatorname{sgn} \{ q_j(s) \} \quad \text{if} \quad |q_j(s)| \geq 1$$

and λ (s) satisfies the equation

$$\lambda(s) = - F^T \lambda(s)$$

and the solution for the above system

$$\lambda(s) = \eta e^{-F^T(s)} \text{ where } \lambda(0) = \eta$$

We now consider a specific second order problem and solve the complete problem by use of phase plane analysis.

Equation of motion is given by

$$\frac{d^2x}{ds^2} = u(s)$$

Putting this equation into Jordan canonical form,

$$\begin{pmatrix} \dot{x}_1(s) \\ \dot{x}_2(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} + \begin{pmatrix} 0 \\ u(s) \end{pmatrix}$$

It is desired to force the plant from an arbitrary state $x_i(0) = \xi_i$ $(i = 1, 2)$ to the origin(0,0) using a minimum amount of fuel

$$F_f = \int_0^T |u(s)| \; ds$$

subject to the constraint $|u(s)| \leq 1$

From necessary conditions we see that u(t) must obey the equations

$$u(s) = 0 \text{ if } |\lambda_2(s)| < 1$$

$$u(s) = -\text{sgn}\{\lambda_2(s)\} \text{ if } |\lambda_2(s)| \geq 1$$

Hence for minimum fuel control $u(t) = 0$ or $u(t) = \Delta = \pm 1$.

To determine the sign of u(t), it is necessary to obtain information about the adjoint system. The solution for the adjoint system,

$$\begin{pmatrix} \dot{\lambda}_1(s) \\ \dot{\lambda}_2(s) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1(s) \\ \lambda_2(s) \end{pmatrix} \quad \lambda_i(0) = \eta_i (i = 1,2)$$

is $\lambda_1(s) = \eta_1$

$$\lambda_2(s) = \eta_2 - \eta_1 s$$

As was pointed out in the previous examples, the relation which exists between x_1 and x_2 and η_1 and η_2 is possibly nonlinear and unknown. For this reason we consider all possible forms of the $\lambda_1(s)$ and $\lambda_2(s)$ to determine the possible control sequences.

There are six possible solutions of $\lambda_2(s)$ and are illustrated in Figure (II-4.2).

A control sequence $\{\Delta, 0, -\Delta,\}$ means that first $u(s) = \Delta$ is applied, then $u(s) = 0$, then $u(s) = -\Delta$ is applied.

From the above figures, the allowable sequences are

$$\{0\}, \{\Delta\}, \{0, \Delta\}, \{\Delta, 0\}, \{\Delta, 0, -\Delta\}$$

To reach the origin with a minimum amount of fuel, one of the above sequences must be used.

ADJOINT SOLUTIONS

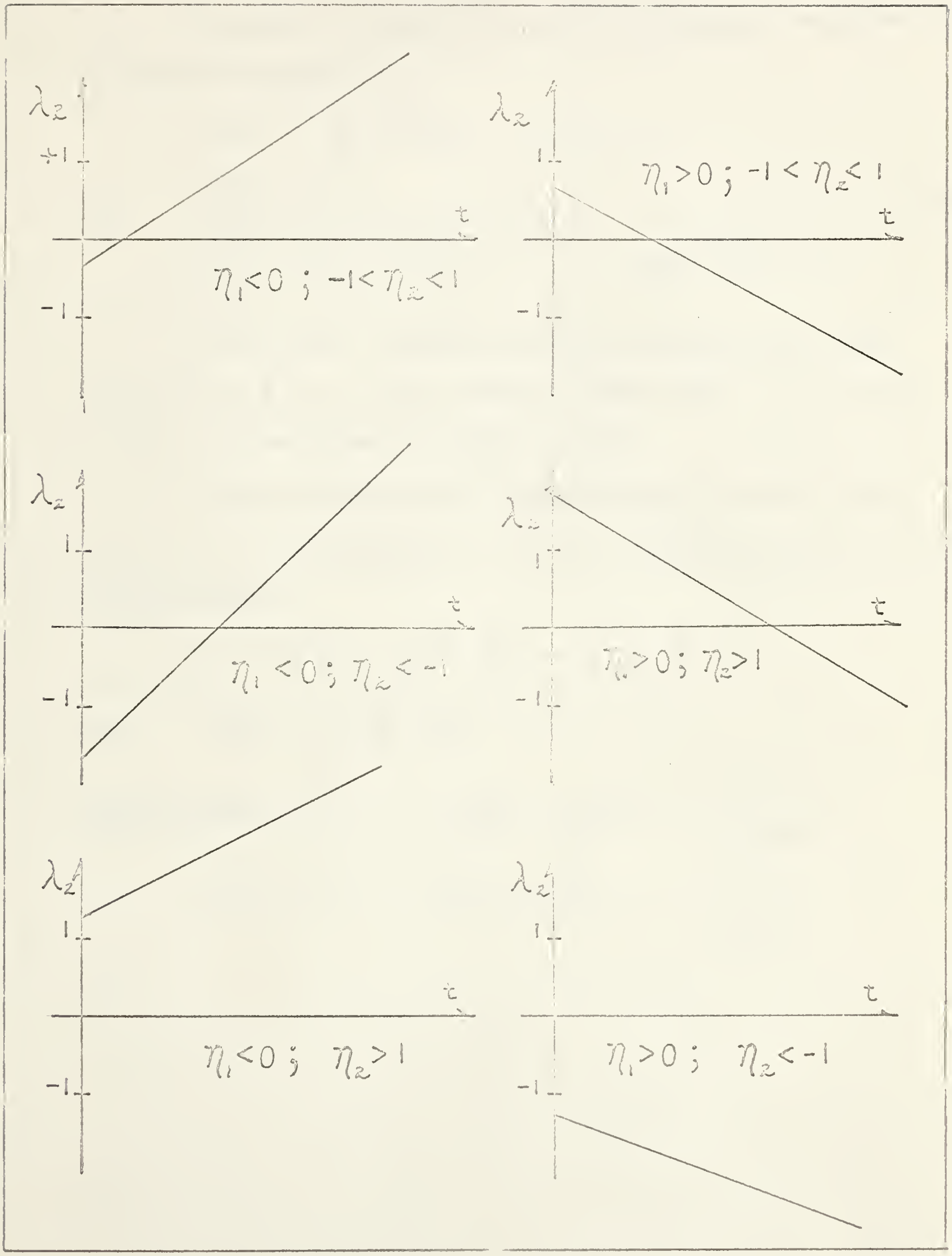


FIG 11 (4-3)

We must now investigate the state solutions. The solutions of the state equations are

$$\left.\begin{aligned} x_1(s) &= \xi_1 + \xi_2 t \\ x_2(s) &= \xi_2 \end{aligned}\right\} \text{for } u(s) = 0$$

$$\text{and} \quad \left.\begin{aligned} x_1(s) &= \xi_1 + \xi_2 t + \tfrac{1}{2}\Delta t^2 \\ x_2(s) &= \xi_2 + \Delta t \end{aligned}\right\} u(t) = \Delta = \pm 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4.21)$$

Let us first consider the plot in the phase plane which forces (ξ_1, ξ_2) to (0,0) using the control $u(s) = 0$, this set contains only one point, the origin itself.

Consider the set of all states in the plane which may be forced to (0, 0) using $u(s) = +1$ or -1. Solving for $x_1(t)$ in terms of $x_2(t)$ we get

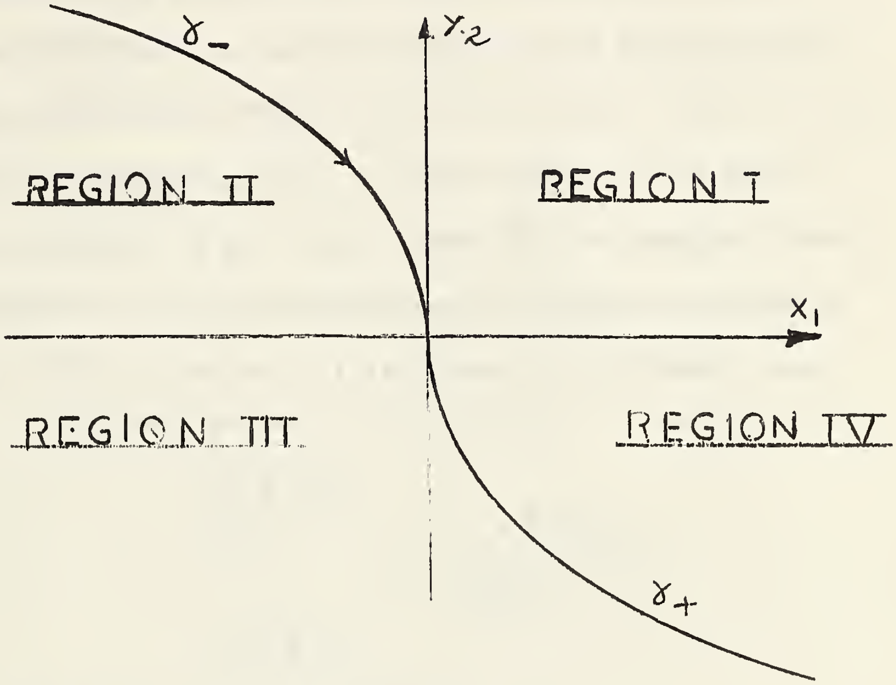
$$x_1 = \xi_1 + \frac{1}{2}\Delta\, x_2^2 - \frac{1}{2}\Delta\, \xi_2^2$$

$$\text{and} \quad x_1(T) = \frac{1}{2}\Delta\, x_2^2$$

$$\text{hence for } u(s) = 1 \qquad x\ (s) = \frac{1}{2}\, x_2^2 \qquad \begin{cases} 0 \leq x_1 \\ x_2 \leq 0 \end{cases}$$

$$u(s) = -1 \qquad x(s) = \frac{1}{2}\, x_2^2 \qquad \begin{cases} x_1 \leq 0 \\ 0 \leq x_2 \end{cases}$$

Denote these sets as γ_+ and γ_- respectively. We now break the remaining plane into four regions as shown below.



These regions are defined as follows,

Region I is the set of points such that $x_2 \geq 0$; $x_I > \gamma_- = -\frac{1}{2} x_2^2$

Region II is the set of points such that $x_2 \geq 0$, $x_I < \gamma_-$

Region III is the set of points such that $x_2 \leq 0$, $x_I < \gamma +$

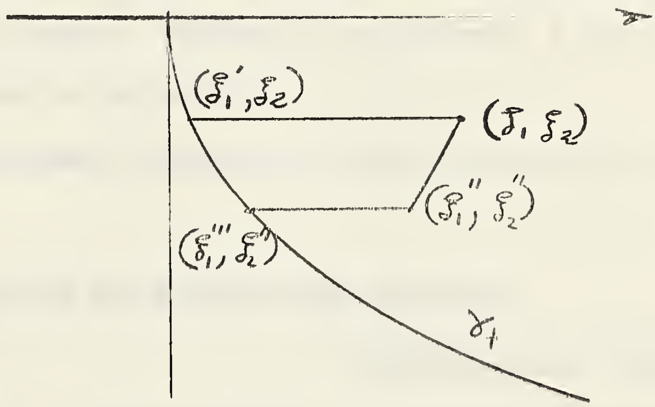
Region IV is the set of points such that $x_I > \gamma_+$, $x_2 < 0$

We now consider the necessary sequence to drive the plant from some initial state (ξ_1, ξ_2) to zero for minimum fuel.

Let (ξ_1, ξ_2) be a point on the curve γ^+. The only control sequence which will force (ξ_1, ξ_2) to the origin is $\{+I\}$ any other sequence cannot force (ξ_1, ξ_2) to the origin. Similarly, the control $\{-1\}$ will force (ξ_1, ξ_2) on γ_- to the origin.

Let us now consider Regions IV. There are several control sequences which could drive (ξ_1, ξ_2) to the origin, it is necessary to determine the one which does so with minimum fuel. The sequences which work are $\{0, +I\}$, $\{-I, 0, I\}$.

For the sequence $\{0, +I\}$ when $u(t) = 0$ is applied $x_2 = \xi_2$ and $x_I = \xi_1 + \xi_2 t$ where ξ_2 is negative, hence x_I will decrease as "t" increases and x_2 will remain constant to (ξ_1', ξ_2). This situation is illustrated in the figure below



Then the application of +I will drive the plant to the origin. In this case we compute the fuel required to be $F = |\xi_2|$

If the control sequence $\{-I, 0, I\}$ is used, application of $u = -I$ will force the state (ξ_1, ξ_2) to a state (ξ_1'', ξ_2'') where $|\xi_2''| > |\xi_2|$. At (ξ_1'', ξ_2'') we apply $u = 0$ which drives (ξ_1'', ξ_2'') to (ξ_1''', ξ_2) on γ_+.

At (ξ_1''', ξ_2'') we apply $u = I$ to drive (ξ_1''', ξ_2'') to the origin. The fuel in this case to reach the origin from (ξ_1''', ξ_2'') is $F_f = |\xi_2''| > |\xi_2|$. Hence the sequence $\{0, +I\}$ is the minimum sequence required.

We can do the same thing for the Region II to get the minimum sequence $\{0, -I\}$.

Region I and III, however, are more interesting and it will be shown that one must first drive $(\xi_1 \xi_2)$ into the adjacent region and then to the origin.

Consider the Region I where the allowable sequences are $\{-I, 0, I\}$ in this case u = -I is applied and drives the point $(\xi_1 \xi_2)$ into Region IV where the sequence $\{0, I\}$ will then drive it to the origin. Similarly the sequence $\{I, 0, -I\}$ is the minimum sequence for Region IV.

The allowable sequences for the various regions are listed below

<u>Optimal Control Law for Minimum Fuel Operation</u>

State Point	Optimal Control Policy
On	Apply $u(t) = I$ to reach the origin
On	Apply $u(t) = -I$ to reach the origin
Region IV	I/ Apply $u(t) = 0$ to reach γ_+ then
	2/ Apply $u(t) = I$ to reach the origin
Region II	1/ Apply $u(t) = 0$ to reach γ_- then
	2/Apply $u(t) = -I$ to reach the origin
Region I	1/ Apply $u(t) = -I$ to just enter Region IV
	2/ Apply $u(t) = 0$ to reach γ_+ then
	3/ Apply $u(t) = I$ to reach the origin
Region III	I/ Apply $u(t) = I$ to just enter Region II
	2/ Apply $u(t) = 0$ to reach γ_- then
	3/ Apply $u(t) = -I$ to reach the origin

Let us now extend the Minimum-Fuel second order system to a third order system and develop a partial solution to illustrate some of the computational difficulties that arise when we extend to higher order systems. To the authors knowledge this problem has not as yet been solved.

The canonical system for this problem is given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} u(t) \\ 0 \\ 0 \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots(4.23)$$

and this system is subjected to the initial conditions (ξ_1, ξ_2, ξ_3)

The adjoint system is given by,

$$\begin{pmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\lambda}_3 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -I \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4.24)$$

and this system is subjected to a set of unknown initial conditions $(\xi_1, \xi_2\ \xi_3)$

From the previous example we see that the necessary conditions for optimal control are given by the equations

$$\begin{aligned} u(t) &= 0 && \text{if} \quad |\lambda_1(t)| < I \\ u(t) &= +\mathrm{sgn}\,\{\lambda_1(t)\} && \text{if} \quad |\lambda_1(t)| \geq I \end{aligned} \quad \ldots\ldots\ldots(4.25)$$

The solution of the adjoint system will provide information on the admissible sequences for an optimal control.

Solutions of the adjoint system are given by

$$\begin{aligned} \lambda_1 &= \eta_3 t^2/2 - \eta_2 t + \eta_1 \\ \lambda_2 &= -\eta_3 t + \eta_2 \\ \lambda_3 &= \eta_3 \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.26)$$

By investigating the possible solutions for $\eta_1(t)$ we find that the allowable sequences are,

$\{0\}$, $\{\Delta\}$, $\{0, \Delta\}$, $\{\Delta, 0\}$, $\{\Delta, 0, -\Delta\}$, $\{\Delta, 0, \Delta\}$, $\{0, \Delta, 0\}$, $\{0, \Delta, 0, -\Delta\}$, $\{\Delta, 0, -\Delta, 0\}$, $\{\Delta, 0 - \Delta, 0, \Delta\}$

We now consider the solution of the system

If $u(t) = 0$

then
$$\begin{aligned} x_1 &= \xi_1 \\ x_2 &= \xi_1 t + \xi_2 \\ x_3 &= \xi_1 t^2/2 + \xi_2 t + \xi_3 \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots(4.27)$$

If $u(t) = \Delta = \pm I$

$$\begin{aligned} x_1 &= \Delta t + \xi_1 \\ x_2 &= \Delta t^2/2 + \xi_1 t + \xi_2 \\ x_3 &= \Delta t^3/6 + \xi_1 t^2/2 + \xi_2 t + \xi_3 \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots(4.28)$$

Or in this case it is more convenient to solve these problems non-parametrically.

Putting the equations in the form

$$dx_1 : dx_2 : dx_3 = u(t) : x_1 : x_2$$

If $u(t) = 0$

then
$$\begin{aligned} x_1 &= \xi_1 \\ x_2^2/2 - \xi_2^2/2 &= \xi_I(x_3 - \xi_3) \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.29)$$

If $u(t) = I$

then
$$\begin{aligned} x_1^2 - \xi_1^2 &= 2 \quad (x_2 - \quad 2) \\ 6\Delta^2(x_3 - \xi_3) &= x_1^3 - \xi_1^3 + 3(\xi_2^2\Delta - \xi_1^2)(x_1 - \xi_1) \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.30)$$

As before we now investigate the curve where $u(t) = 0$ will drive any initial point (ξ_1, ξ_2, ξ_3) to the terminal point(0,0,0)

Therefore it must be the case from equation (4.29) that

$x_1 \ (T) = 0$ hence $\xi_1 = 0$

however, because x_2 and x_3 are also zero at "T" so are ξ_2 and ξ_3.

Hence, the path in the phase plane which forces (ξ_1, ξ_2, ξ_3) to the terminal point (0,0,0) using the control $u(t) = 0$ contains only one point , the origin.

Let us now investigate the curve where $u(t) = + I$ will drive the initial point to the terminal point.

In this case the curve is defined by the intersection of the two surfaces below in the region $x_1 \geq 0$ and $x_3 \geq 0$

$$\left.\begin{aligned} I/2\ x_1^2 &= x_2 \\ I/6\ x_1^3 &= x_3 \end{aligned}\right\} \gamma_+ \quad \text{..........................(4.31)}$$

This situation is illustrated in Fig. (II-4.3)

Similarly we can do the same for $u(t) = -I$

Let us now consider one admissible control to illustrate the method which must be applied to determine the desired control law. Consider the sequence $\{0, \Delta\}$.

Thus we must find conditions under which the curves for $u\ (t) = 0$ intersect the curve γ_+ .

Curves for $u(t) = 0$ are given by the equations

$$x_{1_2} = \xi_1$$

$$x_2 = 2\,\xi_1 x_3 + \xi_2^2 - 2\,\xi_1 \xi_3$$

The intersection of the planes $x_1 = \xi_1$ and the surface $x_2^2 = 2\xi_1 x_3 - 2\xi_1\xi_3 + \xi_2^2$ form the space curve on which $u(t) = 0$ will force the plant. We now find what values of (ξ_1, ξ_2, ξ_3) will cause this curve to intersect γ_+ .

For this to occur it must be the case that $I/2\,\xi_1^2 = x_2$ and $I/6\,\xi_1^3 = x_3$. Hence the following equation must be satisfied

$$\xi_1^4 - 24\,\xi_1\xi_3 + I2\,\xi_2^2 = 0 \quad \text{..........(4.32)}$$

Consider the specific example where the initial point is given by (1, 3, 4 5)

Upon substitution into equation (4.32) we see that this point satisfies the necessary condition. This curve will intersect the curve at the point (I,I/2,I/6) as shown on Fig. (II-4.3). This example is used to illustrate the difficulty in obtaining a complete general solution. The complete solution for this problem is as yet not solved.

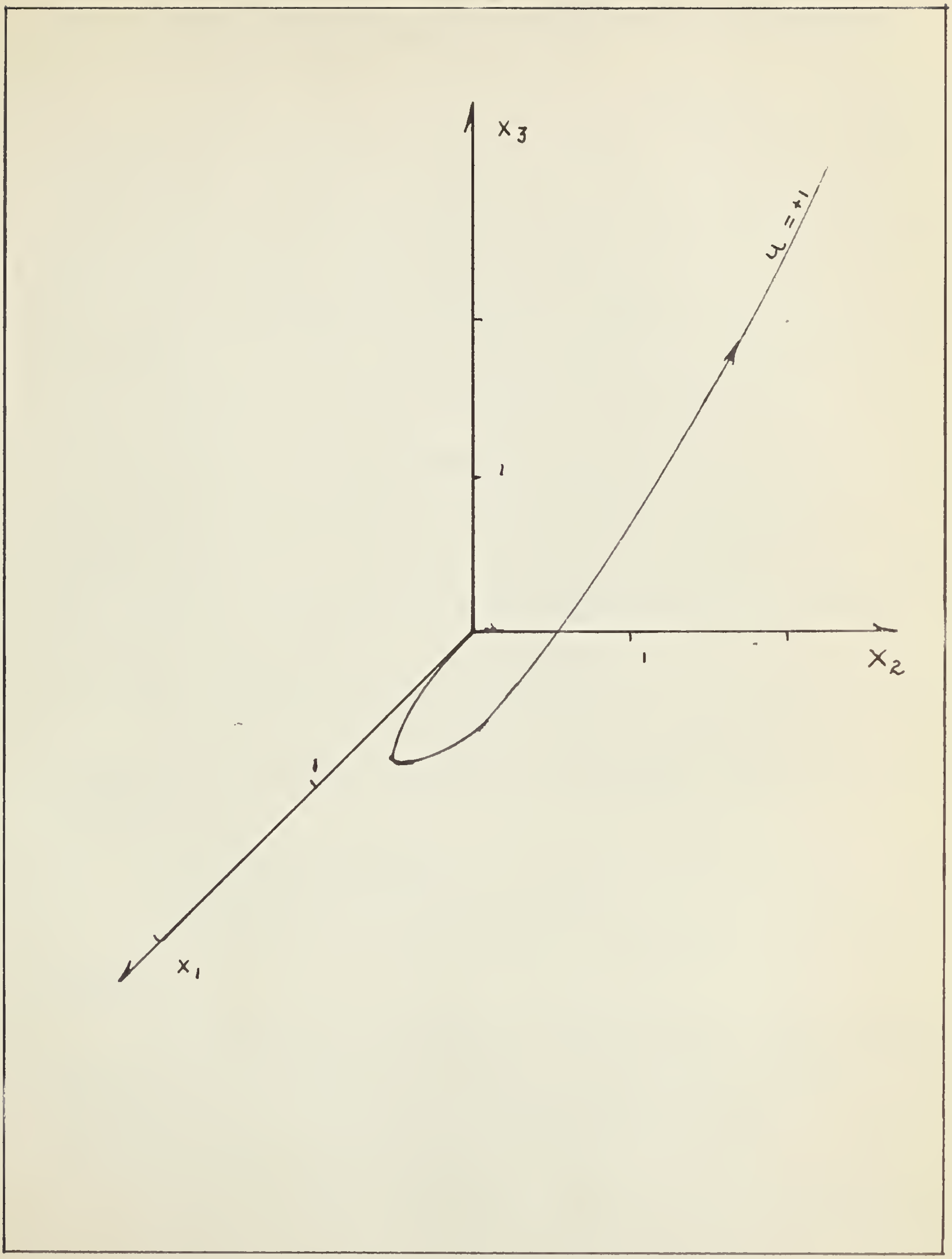


FIG II (4-3)

SECTION III

Part I LINEAR FILTER AND PREDICTION THEORY AND ITS RELATIONSHIP WITH THE LINEAR REGULATOR PROBLEM

Classical results based on Wiener's theory [15, 19] in the field of filtering and prediction have been studied in detail for stationary cases by the use of transform techniques. For stationary problems the synthesis of a filter is done by the method of "Spectorical Factorization". These results where extended to the time domain for the solution of nonstationary problems by Steeg [21] and Shinbrot [20] using impulse response methods. The difficulty in the nonstationary case, once the impulse response has been determined, is to synthesize a filter from this information, this of course is not easy because the synthesis of an integral equation to a system of differential equations is not unique. This problem is sometimes called the identification of the system from its impulse - response matrix. Appendix B .

In this section the classical Wiener - Hopf Equations will be synthesized by a system of differential equations, this is carried out in Appendix (C).

Optimal Filter Problem

The optimal filter problem will be formulated under the following assumptions.

(A_1) Assume that the input is composed of two random signals, one being the desired and the other being the undesired tending to obscure the desired input. The input is given by the equation

$$z(t) = v(t) + H(t)x(t) \qquad \text{(1.1)}$$

where v(t) is the undesired signal (process) and x(t) is the desired signal (message).

(A_2) Assume that the undesired process v(t) is white noise and independent from x(t)

(A_3) Assume that the message x(t) can be generated by passing white noise through a linear system given by

$$\frac{dx}{dt} = F(t)x(t) + G(t)u(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.2)$$

where u(t) is an independent random process (white noise).

If the above assumptions hold and in addition we assume that u(t) and v(t) have identically zero means, the covariances will be given by

$$\begin{aligned} \text{cov.}\left[u(t),u(\tau)\right] &= Q(t)\cdot\delta(t-\tau) \\ \text{cov.}\left[v(t),v(\tau)\right] &= R(t)\cdot\delta(t-\tau) \quad \ldots\ldots\ldots\ldots(1.3) \\ \text{cov.}\left[v(t),u(\tau)\right] &= 0 \end{aligned}$$

(A_4) The measurement of $z(\tau)$ starts at some fixed instant "t_o" of time (which may be $-\infty$), at which time the cov. $\left[x(t_o),x(t_o)\right]$ is known. In the special case when the dynamic system has reached steady - state under the action of u(t), then x(t) is a random function defined by

$$x(t) = \int_{-\infty}^{t} \Phi(t,\tau)G(\tau)u(\tau)d\tau \quad \ldots\ldots\ldots\ldots(1.5)$$

This formula is valid only if the limit as $t_o \longrightarrow -\infty$ exists and is uniformly asymptotically stable.

Under the above assumptions we can now solve the "Optimal Estimation

Problem". Define the optimal estimate of $x(t_1)$ to be $\hat{x}(t_1 \mid t)$, where $t_1 = t$, $t_1 < t$, or $t_1 > t$.

If $t_1 > t$ we have a prediction problem.

If $t_1 = t$ we have a filter problem.

If $t_1 < t$ we have a data smoothing problem. (This problem is not treated in this paper).

The "Optimal Estimation Problem", which is a generalization of well known "Mean-Square Error" used in classical stationary problems, is stated as follows.

Given known values of $z(\tau)$ in the time - interval $t_o \leq \tau \leq t$, find an estimate $\hat{x}(t_1 \mid t)$ of $x(t_1)$ of the form

$$\hat{x}(t_1 \mid t) = \int_{t_o}^{t} A(t_1,\tau)z(\tau)d\tau \quad \ldots\ldots\ldots\ldots(1.6)$$

(where A is an $n \times p$ matrix whose elements are continuously differentiable in both arquments) with the property that the expected squared error in estimating any linear function of the message is minimized:

$$\mathcal{E}\left[x^*, x(t_1) - \hat{x}(t_1 \mid t)\right]^2 = \text{minimum for all } x^* \quad \ldots.(1.7)$$

where x^* is the costate of x see Appendix A .

A special case of the variance estimator of the loss function is the more well known

$$\left\| x(t_1) - \hat{x}(t_1 \mid t) \right\|^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.8)$$

The optimal estimation problem is depicated in Fig. (3-11).

In the above assumptions it was stated that both v(t) and u(t) have identically zero means, using this we now see that the variances estimator of the loss function becomes.

$$\mathcal{E}\left[x^*, x(t_1) - \hat{x}(t_1 | t)\right]^2 = \mathcal{E}\left[x^* \left[x(t_1) - \hat{x}(t_1 \; t)\right]\left[x(t_1) - x(t_1 \; t\right]^T x^*\right] = \|x^*\|^2 \mathcal{E}\, \tilde{x}(t_1 | t)\, \tilde{x}(t_1 | t)^T$$

where $\tilde{x}(t_1 | t)$ is given by

$$\tilde{x}(t_1 | t) = x(t_1) - \hat{x}(t_1 | t)$$

and is called the optimal error function.

Solving the optimal estimation problem the classical Wiener problem is formulated.

Wiener - Hopf Equation

A necessary and sufficient condition for $\left[x^*, \hat{x}(t_1 | t)\right]$ to be a minimum variance estimator of $\left[x^*, x(t_1)\right]$ for all x^*, is that the matrix function $A(t, \mathcal{T})$ satisfy the relation

$$\text{cov.}\left[x(t_1), z(\mathcal{T})\right] - \int_{t_0}^{t} A(t_1, \mathcal{T})\,\text{cov.}\left[z(\mathcal{T}), z(\sigma)\right] d\mathcal{T} = 0 \quad ..(1.9)$$

or equi alently

$$\text{cov.}\left[\hat{x}(t_1 | t), z(\mathcal{T})\right] = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.10)$$

A Corrollqry from the Wiener - Hopf Equation is

$$\text{cov.}\left[\tilde{x}(t_1 | t), \hat{x}(t_1 | t)\right] = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.11)$$

In Appendix C the Wiener-Hopf Equations are converted into a system of differential equations. A summary of these results are as follows.

(1) The canonical form of the optimal filter.

The optimal estimate $\hat{x}(t | t)$ is generated by a linear dynamical system of the form

$$\frac{d\hat{x}(t | t)}{dt} = F(t)\hat{x}(t | t) + K(t)\tilde{z}(t | t) \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{I}$$

where $\tilde{z}(t | t) = z(t) - H(t)\hat{x}(t | t)$

and the initial state of equation I is given by $\hat{x}(t_o | t_o)$ which in turn must be equal to zero.

MESSAGE PROCESS FOR OPTIMAL FILTER

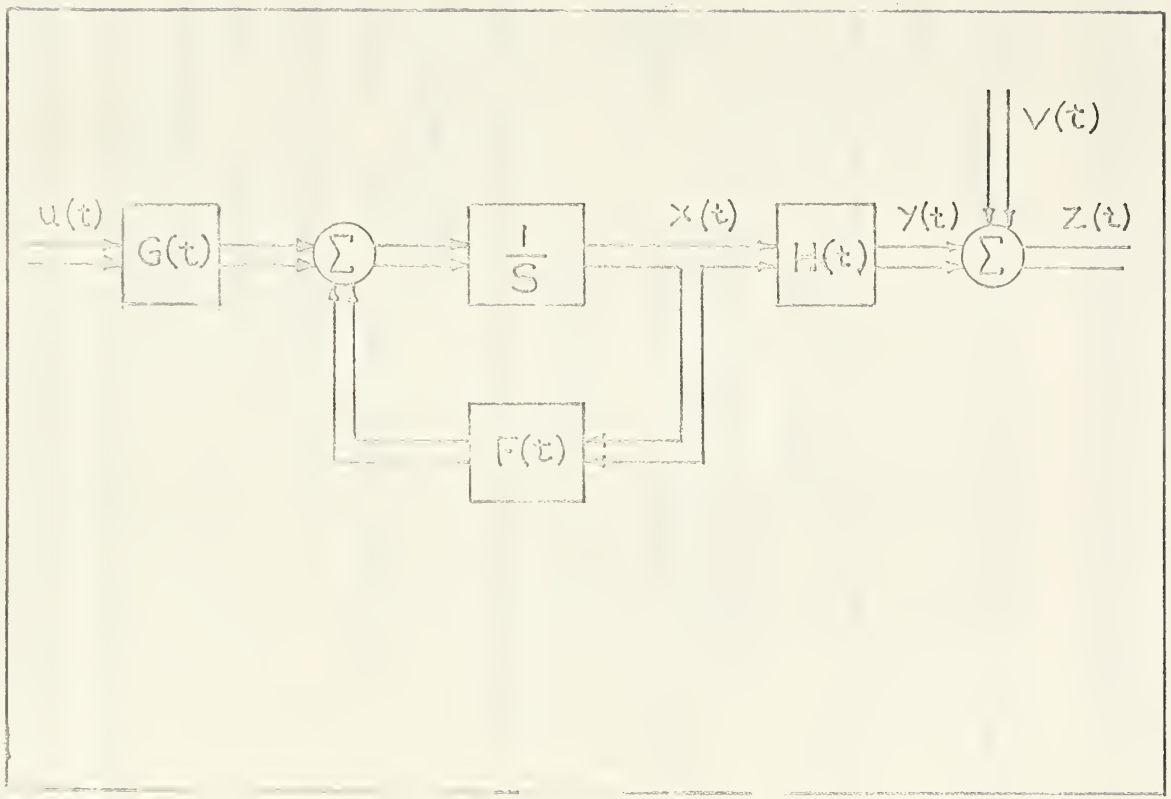


FIG. III (1-1)

For Optimal Extrapolation (prediction) we add to equation I the relation

$$\hat{x}(t_1 \mid t) = \Phi(t_1, t)\hat{x}(t \mid t) \quad \text{for all } t_1 \geq t \quad \ldots\ldots.(1.12)$$

The optimal filter which depicates the above equations is shown in Fig. (III-1.2).

(2) The canonical form for the dynamical system governing the optimal error.

The optimal error defined by the Optimal Estimation Problem, is given by $\tilde{x}(t \mid t) = x(t) - \hat{x}(t \mid t)$ ....(1.13)

The canonical form is given by equation II.

$$\frac{d\tilde{x}(t \mid t)}{dt} = F(t)\tilde{x}(t \mid t) + G(t)u(t) - K(t)\left[v(t) + H(t)\,\tilde{x}(t \mid t)\right] \quad \ldots\ldots\ldots\text{II}$$

This situation is depicated in Fig. (III-1.3).

(3) Optimal gain.

Define $P(t) = \text{cov.}\left[\tilde{x}(t \mid t), \tilde{x}(t \mid t)\right]$ ....(1.14)

Then the optimal gain is given by

$$K(t) = P(t)H^T(t)R^{-1}(t) \quad \ldots\ldots\ldots\text{III}$$

(4) Variance equation

The solution of P(t) is of central importance because the form of the optimal filter is known (see Fig.III-1.2)and the only unknown is the optimal gain which follows immediately once P(t) is known. The solution for P(t) is given by the nonlinear Riccati equation

$$\frac{dP(t)}{dt} = F(t)P(t) + P(t)F^T(t) - P(t)H^T(t)R^{-1}(t)H(t)P(t) + G(t)Q(t)G^T(t) \quad \ldots\ldots\ldots\text{IV}$$

OPTIMAL FILTER

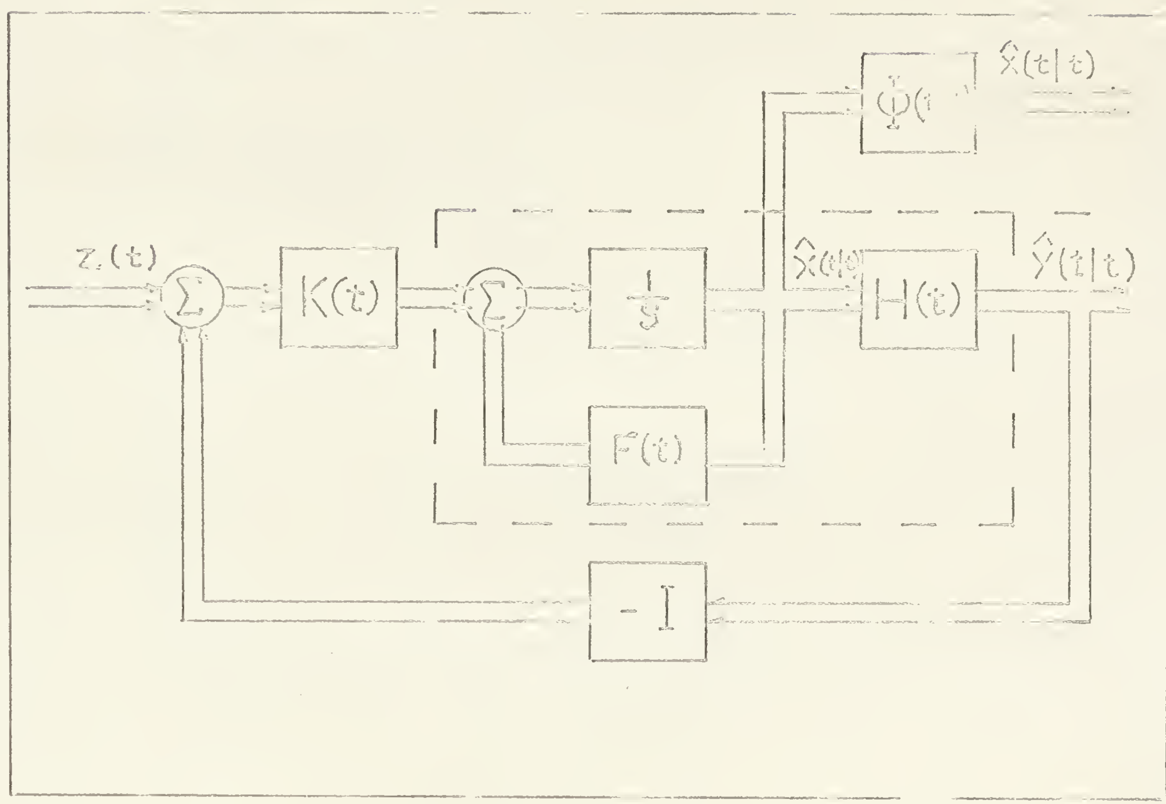


FIG. III (1-2)

OPTIMAL ESTIMATION ERROR

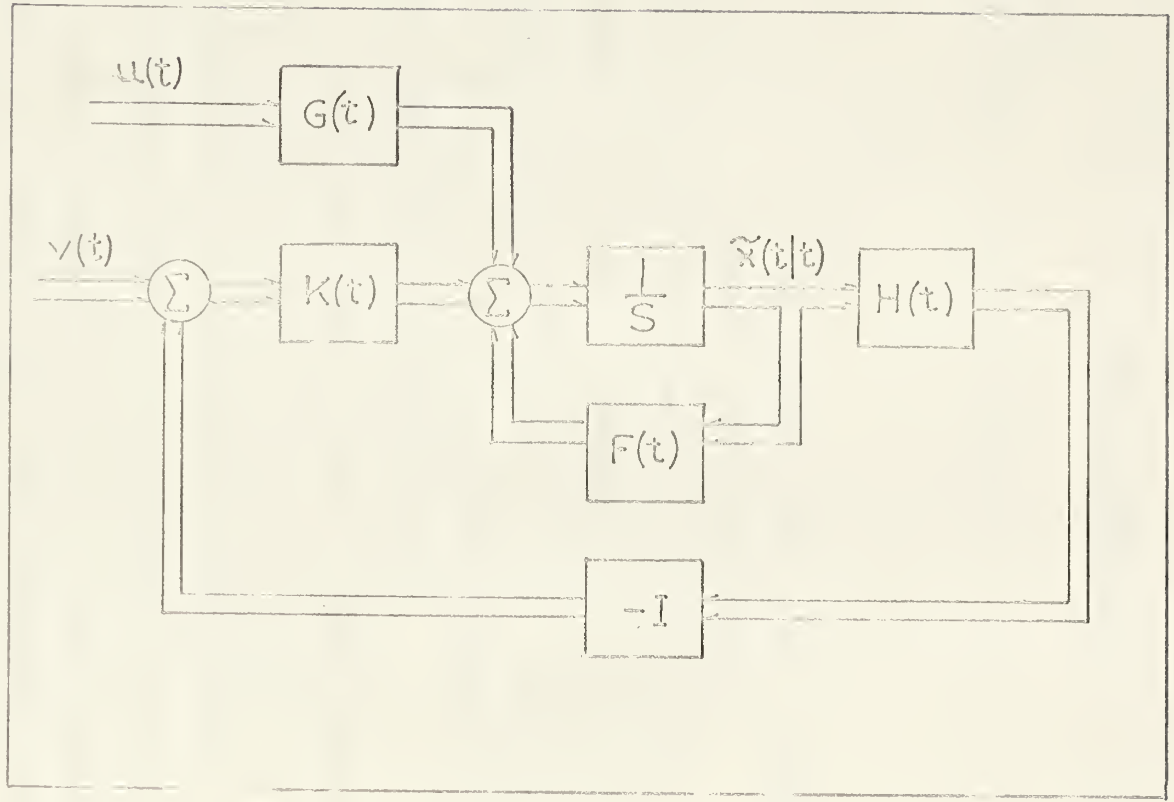


FIG. III (1-3)

where a solution of IV will exist and be uniquely determined for all $t \quad t_o$ by the specification of

$$P(t_o) = P_o = \text{cov.}\left[\tilde{x}(t_o \mid t_o), \tilde{x}(t_o \mid t_o)\right] \quad \ldots\ldots\ldots\ldots(1.15)$$

which because

$$\tilde{x}(t_o \mid t_o) = x(t_o) - \hat{x}(t_o \mid t_o) = x(t_o) = x_o$$

becomes

$$P_o = \text{cov.}\left[x_o, x_o\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.16)$$

because for all $\left| t - t_o \right|$ sufficiently small, given a nonnegative definite matrix P_o, IV satisfies a Lipschitz condition.

The above results are summarized in the following theorem.

Theorem (III-1). Under the assumptions $\left(A_1 - A_4\right)$, the solution of the optimal estimation problem with $t_o < -\infty$ is given by relations (I-IV). The solution P(t) of IV is uniquely determined for all $t > t_o$ (as explained above by the specification of P_o.

A difficulty in finding a solution for P(t) occurs when the "steady state solution" of P(t) is treated. The steady state situation solution occurs when $t_o \rightarrow -\infty$, physically this means that an infinite past of the variance is known. If we assume that steady state is reached then x(t) is given by equation (1.5), but this equation holds only if the message model is asymptotically stable, hence Theorem (III-1) no longer applies and some additional conditions must be imposed on the message model.

In addition to the fact that the message model may not be stable is the fact that H(t) may be singular.

The following Theorem gives two sufficient conditions for the steady state problem to be meaningful. The first is a natural consequence of the definition for x(t) to be in a steady state condition. The second is a weaker condition developed in Section I Part II.

Theorem (III-2). Denote the solution for P(t) as

$$P(t) = \prod(t;P_o,t_o) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.17)$$

then $\lim\limits_{t_o \to -\infty} \prod(t;0,t_o) = \overline{P}(t)$

This limit will exist if either the following conditions hold

(i) The message model is uniformly asymptotically stable; or

(ii) the message model is "completely observable"

Proof: The proof of this theorem follows by dualizing the result (3.20) for the regulator problem.

The matrix $\overline{P}(t)$ is the covariance matrix of the optimal filter described by equation I when the following conditions hold.

(i) An arbitrarily long record of past measurements is available

(ii) The initial state $x(t_o)$ is known exactly.

From equation IV we see that if the matries for the message are constants then so will P(t) be a constant. Because the matries for the message process are constant this in no way implies that P(t) will be a constant, but it does imply that if a limit exists then it will be constant. If the matries for the message process are constant then in the limit the problem reduces to the classical stationary Wiener problem. (subject to conditions which will be shown later).

Further the variance equation suggests a simple method of determining the optimal gain for steady state cases with constant matries, this is obtained by simply setting $dP(t)/dt = 0$ because $P(t)$ is constant. The problem then reduces to one of solving a system of nonlinear algebraic equations. A difficulty arises out of this method however, that is the solution of algebraic equations because they are quadratic in nature are not in general unique. Even if $P(t)$ is required to be nonnegative definite, the algebraic equations may fail to have a unique solution, consider the following example

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad G = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad H = (0 \quad 1)$$

from which $\overline{P}(t)$ can be calculated to be, for $r_{11} = 1$, $q_{11} = 1$

$$2P_{12} + 1 = P_{12}^2 \; , \quad P_{11} + P_{22} - 1 = P_{12}P_{22}$$

$$2P_{12} + 1 = P_{22}^2$$

Solving these equations one obtains the following solutions for $\overline{P}$

$$P_1 = \begin{pmatrix} \sqrt{2} - 1, & -\sqrt{2} + 1 \\ -\sqrt{2} + 1, & \sqrt{2} - 1 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 3 + \sqrt{2}, & 1 + \sqrt{2} \\ 1 + \sqrt{2}, & 1 + \sqrt{2} \end{pmatrix}$$

$$P_3 = \begin{pmatrix} \sqrt{2} + 5, \\ \sqrt{2} + 1, \end{pmatrix} \begin{pmatrix} \sqrt{2} + 1 \\ \sqrt{2} + 1 \end{pmatrix} \qquad P_4 = \begin{pmatrix} -1 - \sqrt{2}, & 1 + \sqrt{2} \\ 1 + \sqrt{2}, & -1 - \sqrt{2} \end{pmatrix}$$

and we see that P_1 is nonnegative definite with P_2 being positive definite, while P_4 is nonpositive definite and P_3 is negative definite. Thus we see that two nonnegative definite solutions for P exist.

But we see that although the message model is observable it is not controllable. Thus as is shown in Theorem (3) the optimal filter may not be asymptotically stable. To determine which of the $\overline{P}$'s actually is the limit of equation IV the solution of IV can be attempted and then take its limit, this, however, is a great deal of work involving finding the roots of a fourth order polynomial with two complex roots. To check whether or not the message process is stable we calculate its transition matrix to be

$$\exp.\ \left[F(t - t_o)\right] = \begin{pmatrix} \cosh(t-t_o) , & \sinh(t-t_o) \\ \sinh(t-t_o) , & \cosh(t-t_o) \end{pmatrix}$$

from which we see that (for an initial x_o of 1,1) that the message process is unstable.

Before we can proceed with the problem of determining what additional condition must be imposed on a system for the Wiener problem to be meaningful we dualize the stability theorem for the regulator problem.

Theorem (III3): Assume that the model of the message process is

(i) uniformly completely observable;

(ii) uniformly completely controllable;

(iii) $\beta_1 \leq \| Q(t) \| \leq \beta_2$, $\beta_3 \leq \| R(t) \| \leq \beta_4$ for all t;

(iv) $\| F(t) \| \leq \beta_5$.

Then the following is true:

(i) The optimal filter is uniformly asymptotically stable;

(ii) Every solution $\Pi(t;\, P_o, t_o)$ of the variance equation(IV) starting at a symmetric nonnegative matrix P_o converges to $\bar{P}(t)$ as $t_o \to \infty$

Thus we see that if the message process is completely controllable and completely observable then the filter is asymptotically stable, and a limit to the variance equation exists as was shown in Theorem (III 2). The problem now is when does the set of algebraic equations obtained by setting the right-hand side of IV equal to zero for P(t), give the same limit as for IV.

An immediate consequence of the above Theorems (2) and (3) is the following result.

If we assume that both Theorem (2) and (3) hold and that the matries F, G, H, Q, R, are constants

Then when $t_o = -\infty$, the solution of the estimation problem is obtained by setting the right-hand side of IV equal to zero and solving the resulting set of quadratic equations. That solution which is nonnegative definite is equal to $\bar{P}$.

To see this, observe from Theorem (2) that the limit of IV $\left[\bar{P}(t)\right]$ exists and is constant. By Theorem (3), all solutions of IV starting at the nonnegative matrix P_o converges to $\bar{P}$. Hence if P is found for which the right-hand side of IV vanishes and if this matrix is nonnegative definite, it must be identical with $\bar{P}$.

There are cases when the conditions on Theorem (3) do not hold, i.e., the filter may not be uniformly asymptotically stable, but we still wish to solve the problem. This case is shown in example (III - 2).

and example (3). It is required to select which of the solutions for P gives the limit of equation IV. To do this consider scalar function.

$$\mathcal{E}\left[x^*,\tilde{x}(t\mid t)\right]^2 = \|x^*\|^2_{\overline{P}(t)}$$

which in Theorem (III 3) was shown to be a Lyapunov function, hence by Lyapunov's Theorem (see Appendix B) if $\overline{P}(t)$ is positive definite of all $x^* = 0$ then the system is stable. Thus we can use this criterion to determine which of the solutions are the actual limit $\overline{P}(t)$ to equation IV.

Returning to example ; we see that the solution to equation IV is given by P_1 because it is nonnegative definite, the positive definite solution cannot exist because this implies that the optimal filter is uniformly asymptotically stable, which in this case is not so.

The only unknown quantity yet to find is the cov. $\left[x(t_o),x(t_o)\right]$ This may be specified as part of the problem. Or in the case when we assume that the message model has reached steady state the cov $\left[x_o,x_o\right]$ is given by

$$S(t) = \text{cov.}\left[x,x\right] = \int_{-\infty}^{t} \Phi(t,\tau)G(\tau)Q(\tau)G^T(\tau)\Phi^T(t,\tau)\,d\tau \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1.18)$$

provided the model is asymptotically stable.

This expression can easily be reduced to the differential equation

$$\frac{dS}{dt} = F(t)S(t) + S(t)F^T(t) + G(t)Q(t)G^T(t) \;\ldots\ldots(1.19)$$

in the case when F, G, Q are constants this equation can be evaluated by equating the right-hand side to zero.

EXAMPLE PROBLEM III - 1

To illustrate the theory developed for the stationary case, consider the following example. This example was also solved by Lee( )using Spectorical Factorization.

The message power density spectrum is given by $\dfrac{a^2}{b^2w^2+1}$; the noise is white with power density spectrum c^2.

Because the problem is stationary, we set the right-hand side of the variance equation equal to zero, resulting in a quadratic equation for P

$$P_{11}^2 - 2r_{11} f_{11} P_{11} - f_{11} q_{11} g_{11}^2 = 0$$

let $g_{11} = h_{11} = 1$ and solve for P_{11} to get

$$P_{11} = r_{11} \left(f_{11} \mp \sqrt{f_{11}^2 + \frac{q_{11}}{r_{11}}} \right)$$

because P_{11} is positive definite (or at least nonnegative definite), the positive sign must hold, hence

$$P_{11} = r_{11} \left(f_{11} + \sqrt{f_{11}^2 + \frac{q_{11}}{r_{11}}} \right)$$

Now using the equation for optimal gain we get

$$k_{11} = f_{11} + \sqrt{f_{11}^2 + \frac{q_{11}}{r_{11}}}$$

The transfer function of the filter can be found by taking the Laplace transforms for Equation I (Note - the initial state of the optimal filter is zero).

The transfer function is given by

$$\frac{k_{11}}{s - f_{11} + k_{11}}$$

To compare this with the filter obtained by Lee, we must find the relationships between a, b, c and k_{11}, f_{11}, and q_{11}.

Using the standard form of input, output power relations for a linear system H(s)

$$\Phi_{oo}(s) = H(-s)\,H(s)\,\Phi_{ii}(s)$$

$$\frac{a^2}{b^2\omega^2+1} = \left(\frac{a}{b}\right)^2 \frac{1}{\omega^2+\frac{1}{b^2}} = \frac{q_{11}}{\omega^2+f^2}$$

From which we see that $q_{11} = \left(\frac{a}{b}\right)^2$ and $f_{11} = \frac{1}{b}$ also $r_{11} = c^2$

Therefore

$$k_{11} = \frac{1}{b} + \sqrt{\left(\frac{1}{b}\right)^2 + \left(\frac{a}{bc}\right)^2}$$

$$= \frac{1}{b} + \sqrt{\frac{a^2 + c^2}{bc}}$$

The transfer function for the optimal filter is

$$\frac{\frac{1}{b} + \sqrt{\frac{a^2 + c^2}{bc}}}{j\omega + \sqrt{\frac{a^2 + c^2}{bc}}}$$

This compares with the corrected optimal filter obtained by Lee (16 , Pg401)

Note: An error in the problem stated by Lee occurs upon substitution of A (bottom page 399) into (29) page 401).

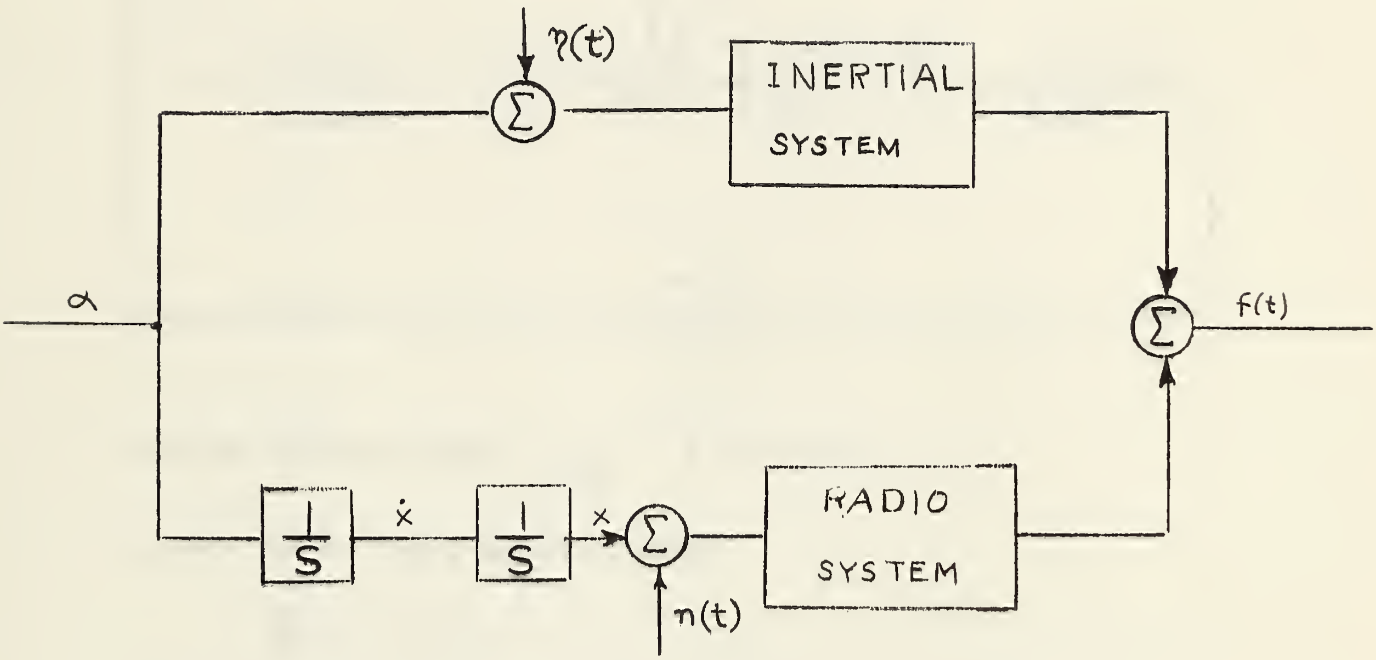
EXAMPLE PROBLEM III - 2

Example of Time Varying Systems

For constant accelerating missile

$\ddot{x}(t) = \alpha$ $\qquad \dot{x}(t) = \alpha t$

and $x(t) = \dfrac{\alpha t^2}{2}$

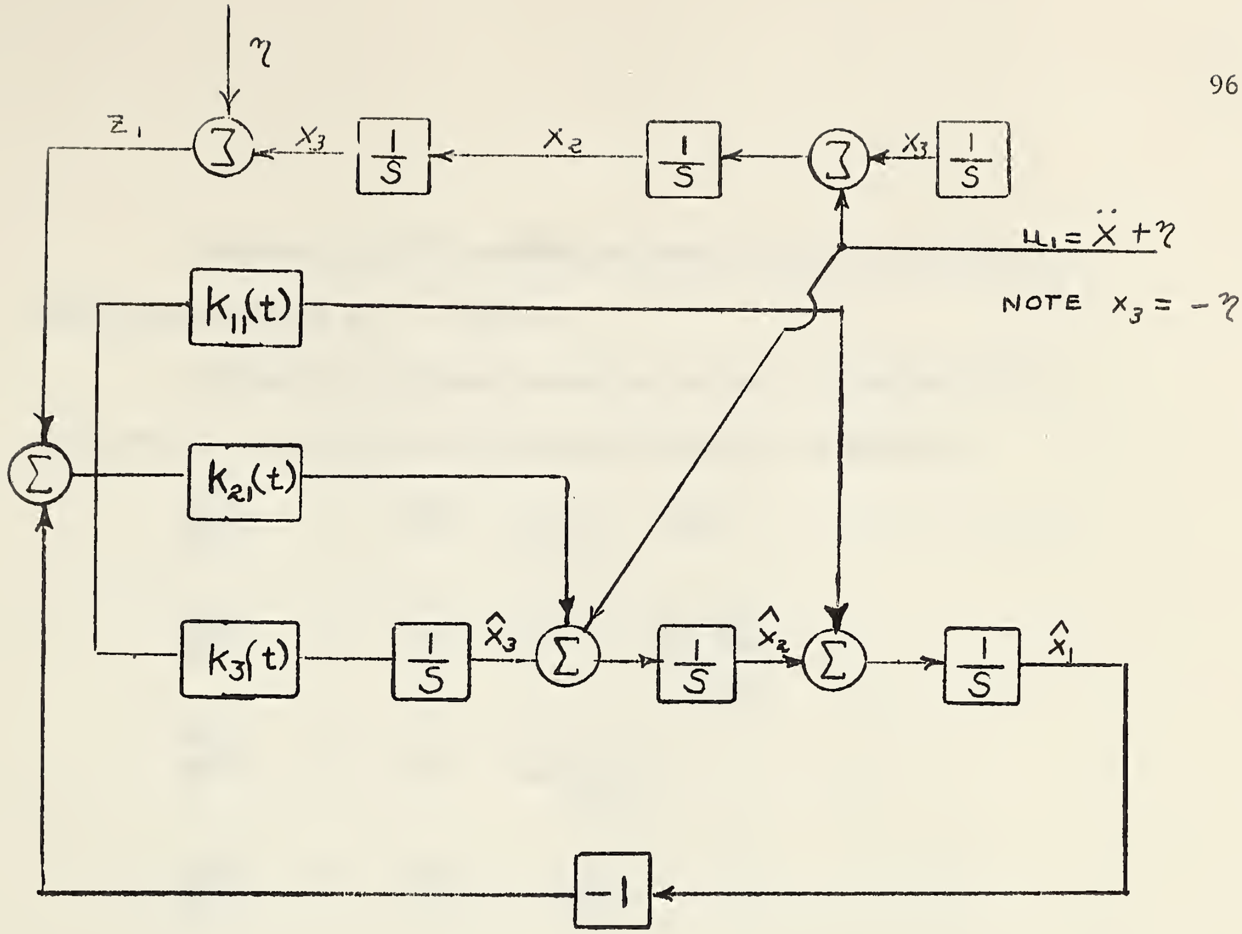


we wish to minimize the error between $\left[x(t) - f(t)\right]^2$

Converting to the message process technique, where n(t) is the radio noise and is an independent white noise source. Using the notation of Section III, cov. $\left[n(t), n(t)\right] = r \cdot \delta\,(t-T)$

The inertial noise is just a constant, and concentrated at zero frequency given by autocorrelation function $\Phi_{nn}(t,T) = Ni$

Because x is known exactly, we can forward it directly into the filter.



From the message process ; $\frac{dx}{dt} = F\,x + Gu(t)$,

we have the following canonical form

$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = \ddot{x} = u_1 - \eta = u_1 + x_3$$

$$\frac{dx_3}{dt} = 0$$

Rewriting the above equations in Jordon's canonical form, results in the following equations.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u_1$$

The output of the message process "y(t)" is related to x(t) by the matrix H = (1,0,0.)

Solving the variance equation, given by Equation IV in Section III, results in the following system of equations

$$\frac{dP_{11}}{dt} = 2P_{12} - \frac{1}{r_{11}} \quad P_{11}^{\ 2}$$

$$\frac{dP_{12}}{dt} = P_{22} + P_{13} - \frac{P_{11}\ P_{12}}{r_{11}}$$

$$\frac{dP_{13}}{dt} = P_{23} - \frac{P_{11}\ P_{13}}{r_{11}}$$

$$\frac{dP_{22}}{dt} = 2P_{23} - \frac{P_{12}^{\ 2}}{r_{11}}$$

$$\frac{dP_{23}}{dt} = P_{33} - \frac{P_{21}\ P_{13}}{r_{11}}$$

$$\frac{dP_{33}}{dt} = P_{13}^{\ 2}$$

It is now necessary to determine the initial state $P(t_o)$. To do this, consider the equation given in Section III denoted by 1.19. For clarity, we repeat equation 1.19 below,

$$\frac{d}{dt} \quad S(t) = FS + SF^T + GQG^T$$

Given that the initial states $x_1 = x_2 = 0$, $x_3 = -\eta$ results in all but S_{33} being zero. Using this we arrive at the following equation

$$\frac{d}{dt}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S_{33} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S_{33} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S_{33} \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

from which we see that S_{33} is given by the constant N_1.

Using the above results we can now form the initial state $P(t_o)$ which is given by

$$P(t_o) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_i \end{pmatrix}$$

Solutions of the nonlinear Riccati equation are found by considering the following 2n canonical equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{r_{11}} & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

Forming the transition matrix for this set of homogenous equations, and setting $t_o = 0$ results in the following matrix

$$\Theta(t,o) = \begin{pmatrix} 1 & 0 & 0 & \frac{t}{r_{11}} & \frac{t^2}{2r_{11}} & \frac{t^3}{6r_{11}} \\ -t & 1 & 0 & \frac{-t^2}{2r_{11}} & \frac{-t^3}{6r_{11}} & \frac{-t^4}{24r_{11}} \\ \frac{1}{2}t^2 & -t & 1 & \frac{t^3}{6r_{11}} & \frac{t^4}{24r_{11}} & \frac{t_o}{120r_{11}} \\ 0 & 0 & 0 & 1 & t & \frac{t^2}{2} \\ 0 & 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

from which the solution for the Riccati equation is given by

$$\Pi(t;P_o;o) = (\Theta_{21} + \Theta_{22}P_o)(\Theta_{11} + \Theta_{12}P_o)^{-1}$$

$$= \frac{\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_i \end{pmatrix}}{\begin{pmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ \frac{1}{2}t^2 & & \\ \hline -\frac{1}{2}t^2 & -t & \end{pmatrix} + \frac{t}{r_{11}}\begin{pmatrix} 1 & t_{12} & \frac{t^2}{6} \\ -\frac{t}{2} & -\frac{t^2}{6} & -\frac{t^3}{24} \\ \frac{t^2}{6} & \frac{t^3}{24} & \frac{t^4}{120} \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_i \end{pmatrix}}$$

$$= \begin{pmatrix} 0 & 0 & \frac{t^2}{2} \\ 0 & 0 & t \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & \frac{t^3N_i}{6r_{11}} \\ -t & 1 & \frac{-t^4N_i}{24r_{11}} \\ -1/2t^2 & -t & 1 + \frac{t^5N_i}{120r_{11}} \end{pmatrix}^{-1}$$

Inverting the second equation, gives us the solution to the Riccati equation, which in turn determines the optimal gain of the time varying filter.

$$P(t) = \left(\frac{r_{11}}{\frac{r_{11}}{N_i} + \frac{t^5}{20}}\right)\begin{pmatrix} \frac{t^4}{4} & \frac{t^3}{2} & \frac{t^2}{2} \\ \frac{t^3}{2} & t^2 & t \\ \frac{t^2}{2} & t & 1 \end{pmatrix}$$

The systems of differential equation which define the optimal filter are given below:

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix} = \begin{pmatrix} \frac{-t^4}{4\beta} & 1 & 0 \\ \frac{-t^3}{2\beta} & 0 & 1 \\ \frac{-t^2}{2\beta} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \frac{t^4}{4} \\ \frac{t^3}{4} \\ \frac{t^2}{2} \end{pmatrix} z_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u_1$$

where β is given by $\frac{t^5}{20} + \frac{r_{11}}{N_i}$.

For solution of variance equation using an analog computer see Fig. ( ).

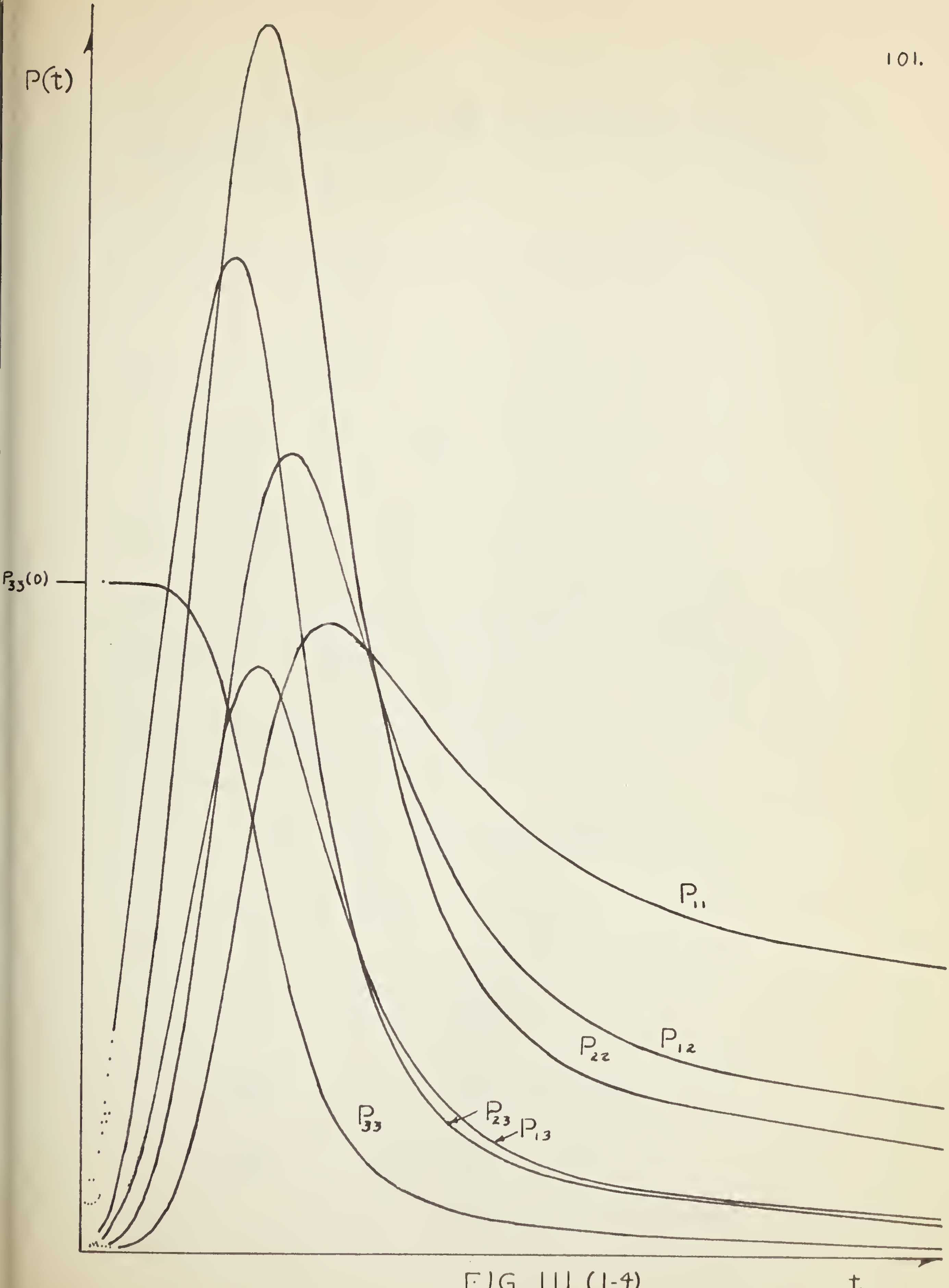
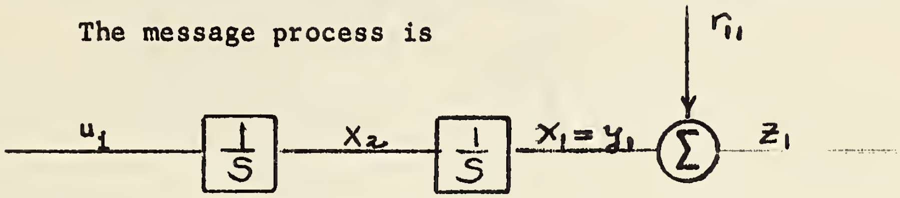


FIG III (1-4)

<u>EXAMPLE PROBLEM III -3</u> (Impulse Response Solution see 21,22)

The message process is



From which we see that

$$\frac{dx_2^2}{dt} = u_1 \text{ and } \frac{dx_1}{dt} = x_2$$

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ u_1 \end{pmatrix}$$

The problem is to determine the position and velocity of a moving particle when the measurements are corrupted by noise.

We assume that the origin of the system occurs at $t = 0$ and that all velocity is a constant

(i,e) $G(t) = 0$

From IV $\quad \frac{dP}{dt} = FP + PFT - PH^T R^{-1} HP$

where $F = \begin{pmatrix} 01 \\ 00 \end{pmatrix} \quad H = (1, 0)$

$$\frac{dPn}{dt} = 2P_{21} - \frac{P_{11}^2}{r_{11}}$$

$$\frac{dP_{12}}{dt} = P_{22} - \frac{1}{r_{11}} P_{11} P_{12}$$

$$\frac{dP_{22}}{dt} = - \frac{1}{r_{11}} P_{12}$$

The matrix coefficient of the Hamiltonian equation is

$$T = \begin{pmatrix} 0 & 0 & 1/r_{11} & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

we can arrive at the transition matrix (i.e.) for $t_o = 0$

$$\Theta(t,o) = \begin{pmatrix} 1 & 0 & \frac{t}{r_{11}} & \frac{t^2}{2r_{11}} \\ -t & 1 & \frac{-t^2}{2r_{11}} & \frac{-t^3}{6r_{11}} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

From

$$P(t) = (\Theta_{21} + \Theta_{22} P_o)(\Theta_{11} + \Theta_{12} P_o)^{-1}$$

The initial conditions are $x_2(0) = \text{constant}$

Therefore

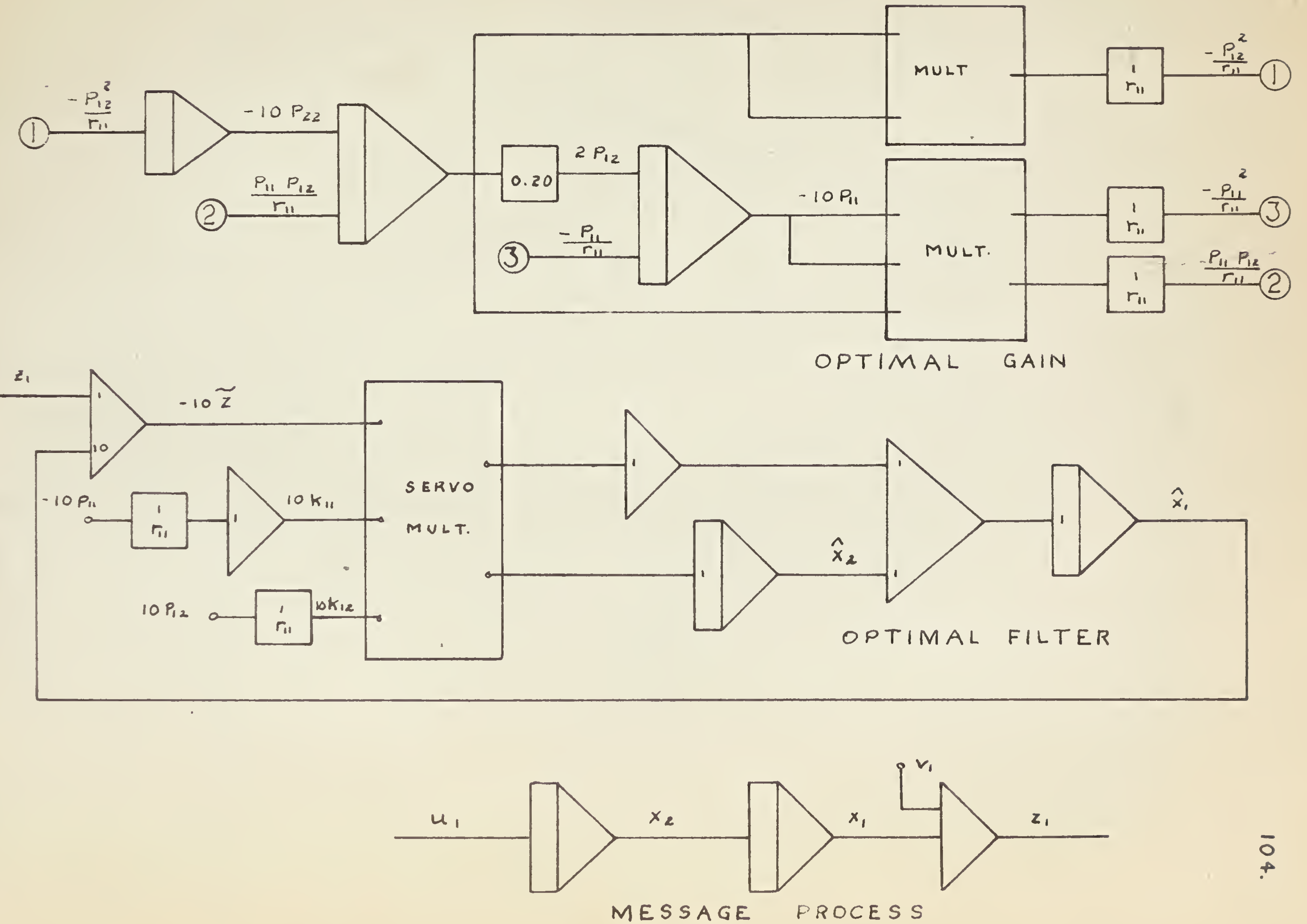
$$P_o = \begin{pmatrix} 0 & 0 \\ 0 & P_{22}(0) \end{pmatrix}$$

The solution for the variance equation is

$$P(t) = \frac{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & P_{22}(0) \end{pmatrix}}{\begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} + \frac{1}{r_{11}} \begin{pmatrix} t & \frac{t^2}{2} \\ \frac{-t^2}{2} & \frac{-t^3}{6} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & P_{22}(0) \end{pmatrix}}$$

$$= \frac{r_{11}}{\frac{r_{11}}{P_{22}(0)} + \frac{t^3}{2}} \begin{pmatrix} t^2 & t \\ t & 1 \end{pmatrix}$$

This problem was solved on the analog computer. The results are compared with the analytic solution in Fig. ( ).



OPTIMAL GAIN
MULT
MULT.
0.20
-10 P22
2 P12
-10 P11
OPTIMAL FILTER
SERVO MULT.
-10 Z̃
10 K11
10K12
-10 P11
10 P12
x̂1
x̂2
z1
MESSAGE PROCESS
u1
x2
x1
v1
z1

FIG III (1-5)

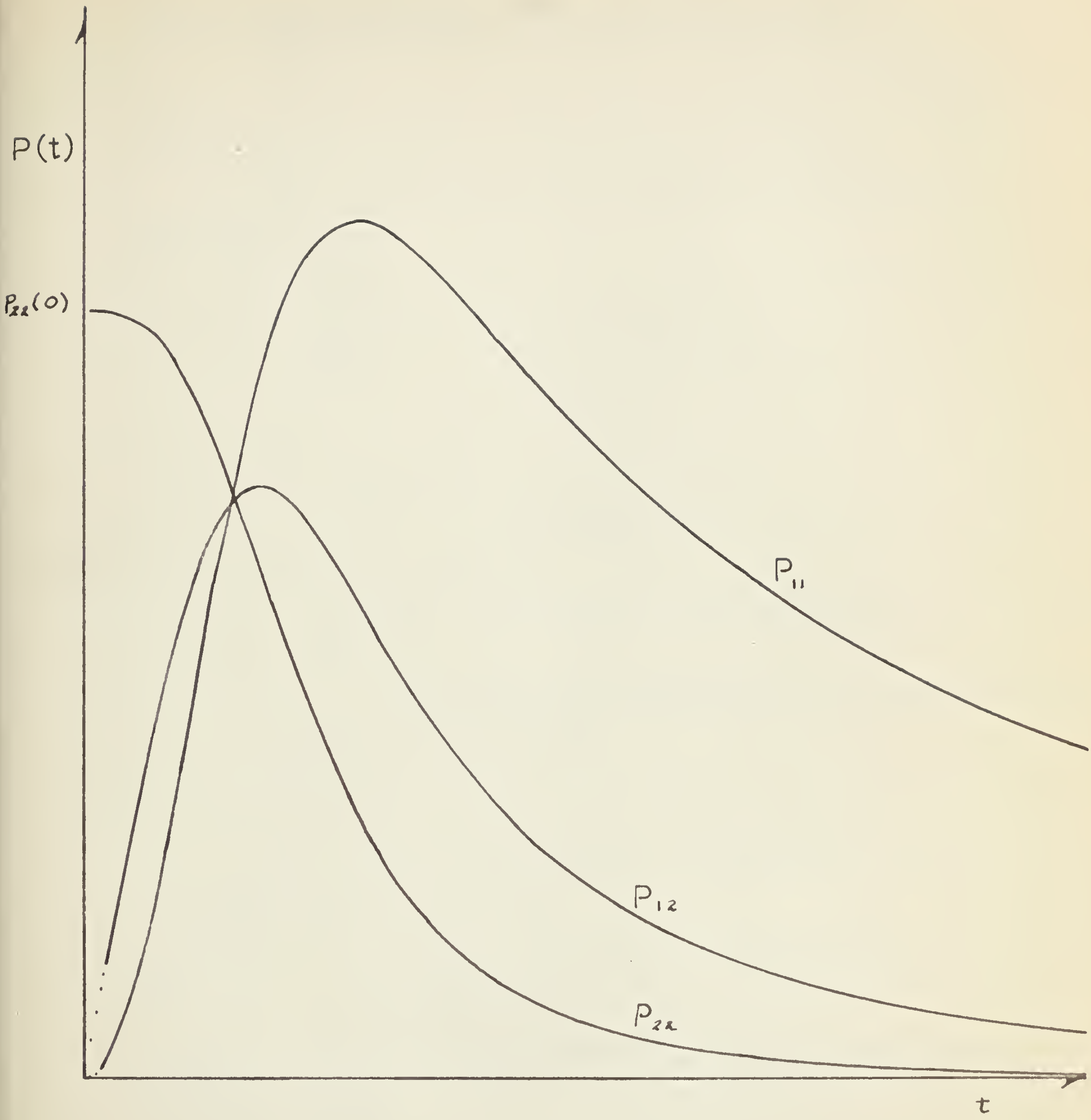


FIG III (1-6)

SECTION III

PART II DUALITY RELATIONS BETWEEN THE LINEAR FILTER AND REGULATOR PROBLEM

To avoid confusion in this part, all equations and arguments for the regulator problem, will be denoted with asterisks.

Under these conditions, the equations for the regulator problem now become

(i) Plant

$$\frac{dx^*(t^*)}{dt} = F^*(t^*)\, x^* + G^*(t^*)u^*(t^*)$$

$$y^* = H^*(t^*)x^*$$

(ii) Control Law

$$u^*(t^*) = -K^*(t^*)x^*(t^*)$$

(iii) Riccati Equation

$$\frac{-dP^*(t^*T)}{dt} = P^*(t^*,T)F^*(t^*) + F^{T*}(t^*)P^*(t^*,T)$$

$$+ H^{T*}(t^*)Qt^*)H^*(t) - P^*(t^*,T)G^*(t^*)R^{-1}(t^*)G^{T*}(t)\ P^*(t^*,T)$$

Under the duality relations,

(a) $-t^* = t$

(b) $F^{T*}(t^*) = F(t), H^{T*}(t^*) = G^*(t), G^{T*}(t^*) = H(t)$

the above equation (iii) becomes

$$\frac{dP^*(t^*,T)}{dt} = P^*(t^*,T)F^T(t) + F(t)P^*(t^*,T) + G(t)Q(t^*)G^T(t)$$

$$-P^*(t^*,T)H^T(t)R^{-1}H(t)P^*(t^*,T)$$

taking the transpose and comparing with the variance equation of the filter problem

$$\frac{dP^{*T}(t^*,T)}{dt} = F(t)P^{*T}(t^*,T) + P^{*T}(t^*,T)F^T(t) + G(t)\ Q\ G^T(t) -$$

$$P^{*T}(t^*,T)H^T(t)R^{-1}H(t)P^{*T}(t^*,T)$$

we see that $P^{*T}(t^*,T) = P(t,t_o)$

Furthermore, by substituting $P^*(t^*T)$ into the optimal control law, it can be shown that K(t) (optimal gain for the linear filter problem is equal to $K^{*T}(t^*)$. Similarly, it can be shown that under duality relations the performance criterion of the linear regulator problem is equivalent to the expected squared error loss function of the linear filter problem.

Hence; we can conclude that under the duality relations, the linear regulator problem (as stated in Section II, Part 3) is equivalent to the linear filter and prediction problem. Thus, all theorems proved for the regulator problem must apply to the filter problem.

SUMMARY AND CONCLUSIONS

In this work, optimal control laws for dynamic systems were synthesized. The controller was considered to be a device which manipulated the inputs to the system, being controlled in accordance with a control law specifying the values of the manipulated variables as a function of the system state and externally generated disturbances acting on the system. The problem dealt with in this thesis was that of finding a control law which makes the system's behaviour in some sense best subject to a pre-assigned performance criterion.

The properties which a controller must have in order to optimize a system, were derived in Section I, Part 1 and 2. It was shown that the control inputs u(t) depended on a set of parameters $\lambda(t)$ and a known input v(t) together with the current state of the system x(t) are the initial conditions of the Euler Lagrange equations. $\lambda(t)$ was to be chosen in such a manner that the Euler Lagrange equations satisfy the predescribed boundary conditions at the future time T. Hence, it was shown that the optimal control law was specified by a solution of a two point boundary-value problem associated with the Euler Lagrange equations.

In the continuous-time case, no general solutions for a nonlinear system have as yet been developed, and for that matter, very few nonlinear problems have been solved in the continuous-time domain. One must solve these problems by using some type of numerical techniques. These methods are beyond the scope of this

report and for this reason, no results for solving nonlinear systems are given. General solutions, however, for linear systems, subjected to a quadratic loss function, have been developed in Section II, Parts 1, 2 and 3. Under the natural boundary conditions, a more elegant technique for solving the linear two-point boundary value problem, was developed in Section II, Part 3. This method, initially suggested by Kalman, develops a differential equation to determine a coefficient matrix, denoted as the Riccati equation in Section II, Part 3, of the linear control law. Computationally, this method, when applicable, provides a much easier solution. Furthermore, if the results are dualized, an analog computer can be used to solve the Riccati equation to determine the control law. The accuracy of using an analog computer for equations of these types, is quite satisfactory, as illustrated in Examples III - 1, 2 and III- 1, 3. If the solution for a stationary problem in a steady state condition is desired, one need only equate the right-hand side of the Riccati equation to zero and solve the resulting quadratic equations for the nonnegative definite solution. Although the system of quadratic equations may be difficult to solve analytically, they are readily solved using an analog computer.

In Section III, Part 4, modern techniques are developed for systems subject to constraints on both the state variable and the control inputs. Some practical problems, using phase plane analysis, are illustrated for second and third order systems with nonlinear controllers. These ideas are an extension to the well-known bang-bang servo problem.

The Wiener-Hopf equations in Section III are reduced to a system of differential equations. This has the advantage in a time varying system, of providing a physically realizable filter. In the steady state condition of a linear stationary problem, the optimal filter can be solved by equating the right-hand side of the variance equations equal to zero, and solving the resulting quadratic equations for the nonnegative definite solution. Computationally, this is much easier, especially for complicated problems, than the more conventional methods (i.e., using transforms and spectorial factorization). Furthermore, the methods presented in Section III are readily applicable to computer solutions whereas, if one used classical techniques, this would be extremely difficult if applicable at all.

It is hoped that the methods presented in this report have contributed to the synthesis of optimal systems and that they will now find practical application in the design of control systems.

APPENDIX A NOMENCLATURE

The nomenclature and symbolism used in this report are consistent with modern terminology in the field of control systems. However, because it is difficult in any given report to find a complete definition of all symbols and nomenclature we enclose this appendix.

Small letters containing the latter part of the alphabet, such as $x(t), y(t)$, etc. will be used to denote vectors. Sometimes an arrow placed above or below the letter is also used However, because in this report no difficulty should be encountered determining whether or not a vector is used, the arrows are dropped. Interminate small letters such as h, g, etc. will be used to denote vector functions, once again the arrows which are used in some reports will be dropped. The components of vectors shall be denoted with subscripts i.e., the components of the n dimensional vector x are $x_1, x_2, \ldots\ldots, x_n$, similarly components of vector functions will be given in the same manner i.e., the vector function f has components $f_1, f_2, \ldots\ldots, f_n$.

Scalar functions will be denoted by capital letters such as $L(x)$; $V(x,u)$, etc. and some small greek letters such as ξ, υ, etc.

Differentiation of a vector is denoted by one of the following methods,

$$\frac{dx(t)}{dt} = \dot{x}(t) = \frac{d}{dt}\begin{pmatrix} x_1(t) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n(t) \end{pmatrix}$$

The derivative of a scalar function is sometimes denoted by the symbol ∇ because the derivative is just the gradient function, and if there is more than one vector argument a subscript on ∇_x is used to denote the variable with respect to which the derivative is taken. This method, however, is awkward and the author prefers to drop the symbol and just use subscripts as shown below.

$$\nabla_x L(x) \quad = \quad L_x(x) \quad = \quad \left(\frac{\partial L}{\partial x_1}, \frac{\partial L}{\partial x_2}, \ldots\ldots, \frac{\partial L}{\partial x_n} \right)$$

The differentiation of a scalar as was pointed out is just the gradient, hence the differentiation of scalar with vestor arguments is vector.

Matries are denoted by capital letters such as A, B etc. and most capital Greek letters i.e., Φ, Ψ, Θ. The transpose of a matrix is indicated as A^T and the inverse by A^{-1}. The symbol a_{ij} denotes the i^{th} row and the j^{th} column of the matrix A. The determinate of A is denoted by det. $|A|$, and adjoint is given by $\tilde{A}$.

The Jacobian matrix (sometimes represented by $J(x)$) gives the differentiation of a vector function. We shall denote vector function differentiation as follows,

$$fx(x,y) \quad = \quad \begin{pmatrix} \frac{\partial f_1}{\partial x_1}, & \frac{\partial f_1}{\partial x_2}, & \ldots\ldots, & \frac{\partial f_1}{\partial x_n} \\ \vdots & \cdot & & \cdot \\ \frac{\partial f_n}{\partial x_1}, & \frac{\partial f_n}{\partial x_2}, & \ldots\ldots, & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \quad \ldots\ldots\ldots(A\text{-}1)$$

Norms.

A norm is a function which assigns to every vector (or point) x in a given Euclidean space a real number $\| x \|$ such that the following axioms are obeyed.

(i) $\| x \| \geq 0$ for all x

(ii) $\| x + y \| \leq \| x \| + \| y \|$ for all x,y

(iii) $\| \alpha x \| = | \alpha | . \| x \|$ for all x and complex constant

(iv) $\| x \| = 0$ implies $x = 0$

If (iv) is missing we say $\| x \|$ is a seminorm.

Some examples of well-known norms are

(a) $\| x \| = \sum_{i=1}^{n} | x_i |$

(b) $\| x \| = \sum_{i=1}^{n} \left(x_i^2 \right)^{1/2}$

This is the standard definition of the distance of a point in Euclidean space from the origin, also called the Euclidean norm.

(c) $\| x \|_A = (x^T A x)^{1/2}$ where A is symmetric and positive definite ; called the generalized Euclidean norm.

(d) $\| x \| = \max_i \left([x_i] \right)$

In explicit calculations, $\| x \|$ always means the Euclidean norm.

This idea can be extended to the norm of a matrix, defined by

$\| A \| = \text{Min } K$ such that $\| Ax \| \leq K \| x \|$

The following inequalities hold

(i) $\| AB \| \leq \| A \| \; \| B \|$

(ii) $\| A^T \| = \| A \|$

(iii) For the Euclidean norm

$$\| A \|^2 = \underset{x}{\mathrm{Max}} \left(x^T A^T A x; \quad x^T x = 1 \right)$$

A matrix A is said to be positive definite [positive semi-definite] if the quadratic form

$$x^T A x = \sum_{ij=1}^{n} x_i \, a_{ij} \, x_j$$

is positive [nonnegative] for all $x \neq 0$.

Further it can be shown that a matrix A is positive-definite [semidefinite] if and only if the following conditions hold:

(i) There is a nonsingular singular matrix B such that $B^T B = A$

(ii) $\lambda_i(A) > 0$ $\left[\lambda_i(A) \geq 0 \right]$ for all i; where $\lambda_i(A)$ denoted the eigenvalue of the $n \times n$ matrix A and are the roots of the polynomial $\det \left(A - \lambda I \right)$.

(iii) All principal minors of A are positive nonnegative .

Class of a System:

A function f(x) is said to be of class C^i if

$$\frac{\partial^i f_1}{\partial x_{j1} \, \partial x_{j2} \cdots\cdots \partial x_{ji}} \quad \text{are all continuous.}$$

A function f(x) is said to be of class D^i if

$$\frac{\partial^i f_1}{\partial x_{j1} \, \partial x_{j2} \cdots\cdots\cdots \partial x_{ji}} \quad \text{are all continuous except on a set of zero area}$$

where, however, both left and right-hand limits exist.

Lipschitz Condition

If a function $f(x,t)$ is Lipschitz condition it must obey the inequality

$$\| f(x,t) - f(y,t) \| \leq k \| x - y \|$$

where k is a positive constant.

Expectance Operator

The expectance operator is defined by $\mathcal{E}$, from which we define the covariance of a function to be [4],

$$\text{cov.} \left[x(t), y(\tau) \right] = \mathcal{E}\, x(t) y^T(\tau) - \mathcal{E} x(t)\, \mathcal{E} y^T(\tau)$$

where $\mathcal{E} x(t)$ is the mean.

The variable assignment used for the control problem throughout this report is as follows:

C Control law matrix

F Matrix of the state transition coefficient in the linear case

f State transition function

G Matrix of coefficients relating control input to state transition in the linear case

g Function specifying terminating conditions

H Matrix specifying system output in the linear case

h Function specifying system outputs in terms of system states

k Control law function

K Matrix of coefficients relating system disturbances to state transition in the linear case

L	Integrand of the performance criterion; i.e., Loss function
M	Observability matrix
m	Number of components of the control input vector
n	Order of the system, the dimensionality of its state vector
P	Matrix of the Riccati equation
p	Number of components of the output vector
S	Impulse response matrix
s	Running time
s	Laplace transform
t	Current time
T	Time at which the control problem terminates
u	Control inputs
V	Performance criterion
V	Lyapunov function
W	Controllability matrix
W	System disturbances
x	State variable
y	System outputs
z	Desired output
τ	Lagrange multiplier function
Φ	Transition matrix of the linear state transition
υ	Specifies performance function at the terminal time
$\mathcal{H}$	Hamiltonian

The variable assignment used for the filter problem throughout this report is as follows:

F	Matrix coefficient of the message
G	Matrix coefficient relating input to the message
H	Matrix coefficient relating output to the message
K	Optimal gain
M	Observability matrix
n	Order of the system, the dimensionality of the message
m	Number of components of the input process
p	Number of components of the output process
P	Covariance matrix of the optimal error
$\overline{P}$	Steady state solution of the Riccati equation
S	Covariance of the initial state of the message
t	Time
t_o	Initial time
t_1	Prediction time
u	Random input signal used to generate the message
v	Undesired random process
x	Desired random process (message)
$\hat{x}(t_1 \mid t)$	Optimal estimate of the message $x(t_1)$
$\tilde{x}(t_1 \mid t)$	Optimal error
x^*	Costate of the vector function x
y	Output of message process
z	Input to optimal filter
$\sum$	Expectance operator
$\prod$	Solution of the variance equation

APPENDIX B

METHOD OF A DYNAMIC SYSTEM

The State Transition Method

Microscopic physical phenomena are commonly described in terms of cause-and-effect relationships. In Newtonian mechanics, the motion of a system of particles is fully determined for all future time by the present position and momentum of the particles and by the present and future forces acting on the system. Future forces can have no effect on what happens at present.

In modern terminology, we say that the numbers which specify the instantaneous position and momentum of each particle represent the <u>state</u> of the system. The state is to be regarded always as an abstract quantity. Intuitively speaking, the state is the minimal amount of information about the past history of the system which suffices to predict the effect of the past upon the future. Further, we say that the forces acting on the particles are the <u>inputs</u> of the system. Any variable in the system which can be directly observed is an <u>output</u>.

The system, in general, will be described by the following nonlinear vector differential equation,

$$\frac{dx(t)}{dt} = f(x(t),u(t);t) \qquad \text{(B-1)}$$

where $x(t)$ denotes an n dimensional vector;

$u(t)$ denotes an m dimensional vector; and

$f(t)$ is a vector function of the vectors $x(t)$ and $u(t)$ as described in Appendix A.

The vector differential equations can be written as an equivalent system of n scalar differential equations,

$$\frac{dx}{dt}_i = f_i(x_1, \ldots, x_n; u_1, \ldots, u_m; t) \quad \ldots\ldots\ldots\ldots\ldots\ldots (B\text{-}2)$$

where ($i = 1,2, \ldots\ldots, n$) and x_i and f_i are components of the vector x and vector function f respectively.

Definition: If the vector function $f(x,u;t)$ is independent of the forcing function $u(t)$ for all t i.e., $u(t)$ 0, then the equation (B-1) becomes

$$\frac{dx(t)}{dt} = f(x(t);t) \quad \ldots\ldots\ldots\ldots\ldots\ldots (B\text{-}3)$$

and the system is said to be free (unforced).

Definition: If the vector function in addition to being free is also independent of t then equation (B-1) becomes

$$\frac{dx(t)}{dt} = f(x(t)) \quad \ldots\ldots\ldots\ldots\ldots\ldots (B\text{-}4)$$

then we say that the system is autonomous.

The output of a system, denoted by $y(t)$, is determined by a set of algebraic equations defined by the vector equation

$$y(t) = h(x(t);t) \quad \ldots\ldots\ldots\ldots\ldots\ldots (B\text{-}5)$$

where $y(t)$ is an p dimensional vector; and

h is a vector function of the vector $x(t)$.

Linear differential equations are expressed in vector form as a system of linear first order equations as follows,

$$\frac{dx(t)}{dt} = F(t)x(t) + G(t)u(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots (B\text{-}6)$$

and

$$y(t) = H(t)x(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots (B\text{-}7)$$

where F(t) is an $n \times n$ dimensional matrix;

G(t) is an $n \times n$ dimensional matrix; and

H(t) is an $n \times p$ dimensional matrix.

The above vector equations are equivalent to a system of scalar equations written as follows:

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} f_{ij}x_j + \sum_{j=1}^{m} g_{ij}u_j \quad \ldots\ldots\ldots\ldots\ldots(B\text{-}8)$$

$$y_i = \sum_{j=1}^{p} h_{ij}x_j \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(B\text{-}9)$$

where $i = 1,2,\ldots\ldots\ldots n$

If we assume that $f(0,0,t) = 0$ then F(t) and G(t) are the Jacobian matrices $\left[\frac{\partial f_i(t)}{\partial x_j(t)}\right]$ and $\left[\frac{\partial f_i(t)}{\partial u_j(t)}\right]$ obtained by the standard linearization method.

The solution of the equation (B-6) is denoted by $\varphi(t;x_o,t_o)$, where x_o is the initial state at time t_o. Then the solution can be expressed by the well-known form.

$$\varphi(t;x_o,t_o) = \Phi(t,t_o)x_o + \int_{t_o}^{t} \Phi(t,\sigma)G(\sigma)u(\sigma)d\sigma \quad \ldots(B\text{-}10)$$

where $\Phi(t,t_o)$ is the so-called transition matrix which obeys the following conditions,

(i) $\frac{d\Phi(t,\sigma)}{dt} = F(t)\,\Phi(t,\sigma)$

subject to the condition that $\Phi(t,t) = I$, where I denotes the unit matrix.

(ii) $\Phi(t_2,t_o) = \Phi(t_2,t_1)\Phi(t_1,t_o)$ for all t_o,t_1,t_2

(iii) $\Phi^{-1}(t,t_o) = \Phi(t_o,t)$ for all t,t_o.

(iv) In the case where F is a constant matrix then the solution for the transition matrix is given by,

$$\Phi(t,t_o) = \exp.\left[t - t_o\right] F = \sum_{k=0}^{\infty} \frac{\left(\left[t - t_o\right] F\right)^k}{k!}$$

The Impulse-Response Matrix

In the filter and predication problem a system of linear differential equations is developed from the well-known Wiener equations. In effect this is the problem of realizing a linear system of differential equations from the systems impulse response, for this reason we include some results of identification theory. The theory of identifying a system of differential equations from an impulse response is a broad and difficult field in itself, for this reason a complete study is not attempted here.

We now define a dynamic system from the point of view of impulse responses. Consider a system which is at rest at time t_o; i.e., one whose inputs and outputs have been identically zero for all $t \leq t_o$. We apply at each input in turn a very sharp and narrow pulse. Ideally, we would take $u_i^j(t) = \delta_{ij}\delta(t - t_o)$, where δ is the Dirac delta function, δ_{ij} is the Kronecker symbol, and $1 \leq i, j \leq m$. We then observe the effect of each vector input $u^j(t)$ on the outputs. The matrix $S(t,t_o) = \left[s_{ij}(t,t_o)\right]$ so obtained is called the impulse response matrix. Since the system was initially at rest we must define $S(t,t_o) \equiv 0$, for all $t < t_o$.

With these conventions, the output of a linear system originally at rest is related to its input by the well-known convolution integral:

$$y(t) = \int_{t_o}^{t} S(t,\tau)u(\tau)d\tau \qquad \text{.......................B-11)}$$

Unfortunately, this definition does not explain how to treat systems which are not initially at rest.

If we suppose that system governed by equations (B-6,7) are initially at rest i.e., $x_o \equiv 0$, then the impulse response and the state transition are related by the following equation.

$$\begin{aligned} S(t,\tau) &= H(t)\Phi(t,\tau)G(\tau) \qquad t \geq \tau \qquad \text{...............(B-12)} \\ &= 0 \qquad\qquad\qquad\qquad t < \tau \end{aligned}$$

From which we see that it is just a matter of substitution to find the impulse response once the terms on the right-hand side are known. If, however, we attempt to determine the system of differential equations from the impulse (experimentally observed) the problem is no longer trivial. Even if we succeed in determining a system which obeys equation (B-11) one must still ask the question, is the system of the lowest possible order. This can be stated in other words as, have we realized an irreducible identification of $S(t,\tau)$. We may now ask ourselves the question "When and how does the impulse matrix determine equations of the system?"

The first condition we consider is stated as follows:

An impulse-response matrix $S(t,\tau)$ is realizable by a finite - dimensional dynamical system (B-6,7) if and only if there exists continuous

matrices P(t) and Q(t) such that

$$S(t,\tau) = P(t)Q(\tau) \quad \text{for all } t, \quad \ldots\ldots\ldots\ldots\ldots (B\text{-}13)$$

This result was assumed to be true by Shinbrot [20] in his development of an optimal time varying filter.

In many cases the impulse response is stationary and can be given by $S(t-\tau)$, which in turn can be represented by its Laplace transform $Z(s) = \mathcal{L}\, S(t)$.

Another very important condition that has been proved, at least the stationary case, [12 23] is stated below.

Knowledge of the impulse-response matrix $S(t,\tau)$ identifies the completely controllable and completely observable part, and this part alone, of the dynamical system which generated it. This part has the smallest dimension among all realizations of $S(t,\tau)$ and is identified by $S(t,\tau)$ uniquely.

Stubberud 23 has using different techniques shown that a reducible system can be transformed into the same form as that stated in Section I, Part 2 to define an uncontrollable system, hence we can conclude on the basic of Stubberud's results that because equivalent forms are used, that controllability and observability must also be conditions which must be obeyed if an irreducible realization is to be obtained.

Stability of a Dynamic System

Optimal systems defined by their optimal control law are no guarantee that the system is stable. Stability must be checked by using one of the following methods depending on the complexity of the problem. If the problem is linear with constant coefficients Routh's Stability Criterion may be used. This theorem is well known in linear control systems and is stated in [ ]. If the system is linear with non-constant coefficients the following theorem may be used. For a formal proof see [ ]. Theorem (B-1). Consider a continuous time, linear dynamic system (B-6).

$$\frac{dx(s)}{ds} = F(s)x(s) + G(s)u(s)$$

subject to the restrictions

(i) $\| F(s) \| \leq c_1 < \infty$ for all s

(ii) $0 < c_2 \leq \| G(s)x \| \leq c_3 < \infty$ for all $\| x \| = I$, all s

Then the following propositions concerning this system are equivalent

(A) Any uniformly bounded excitation

$$\| u(s) \| \leq c_4 < \infty \quad \left| s \geq t_o \right|$$

gives rise to a uniformly bounded response for all $s \geq t_o$

$$\| x(s) \| = \left\| \Phi(s,t_o)x_o + \int_{t_o}^{s} \Phi(s,\sigma)G(\sigma)u(\sigma)d\sigma \right\| \qquad \text{............(B-14)}$$

$$\leq c_5(c_4, \| x_o \|) < \infty$$

(B) For all $s \geq t_o$,

$$\int_{t_o}^{s} \| \Phi(s,\sigma) \| d\sigma \leq c_6 < \infty \quad ; \text{ ....(B-15)}$$

(C) The equilibrium state " x_e " equal zero of the free system is uniformly asymptotically stable;

(D) There are positive constants c_7, c_8 such that, whenever $s \geq t_o$, $\left\|\Phi(s,t_o)\right\| \leq c_7 \exp\left[-c_8(s - t_o)\right]$ ....(B-16)

(E) Given any positive - definite matrix Q(s) continuous in s and satisfying for all $s \geq t_o$

$$0 < c_9 \leq Q(s) \leq c_{10} < \qquad \text{..................(B-17)}$$

the scalar function defined by

$$V(x,s) = \int_s^{\infty} \left\|\Phi(\sigma,s)x\right\|^2_{Q(\sigma)} d\sigma$$

$$= \left\| x \right\|^2_{P(s)} \qquad \text{...................(B-18)}$$

exists and is a Lyapunov function for the free system satisfying the requirements of Lyapunov's Theorem (see), with its derivative along the free motion starting at x,s being

$$\dot{V}(x,s) = -\left\| x \right\|^2_{Q(s)}$$

None of the propositions in Theorem ( ) offers a method of determining whether a given system is uniformly asymptotically stable or not without computing its transition matrix for all s, t_o. For this reason it is generally difficult to determine the stable of a time varying system without doing a great deal of work.

In the case of nonlinear equations the question of determining stability of a system becomes extremely difficult, as of course are the solutions of a nonlinear system. In this paper a direct method of

determining stability suggested by Lyapunov, called the second method of Lyapunov, will be used. Unlike Lyapunov's first method which only give information on asymptotic system about a point the second method can be used to determine various types of stability. The difficulty of the second method will become apparent upon statement of the theorem.

Lyapunov's Second Method

The second method is based on the idea that if one is able to find a suitable continuous scalar function V(x,s) which is positive for $x \neq 0$ and zero for x = 0, then along a trajectary x(s) an inequality of the form $V(x(s)) \leq \delta$ will guarantee a related inequality $x(s) \leq \epsilon$ with $\epsilon \rightarrow 0$ as $\delta \rightarrow 0$. The quantity V(x) is then a measure of the smallness of x.

Theorem (B-2)

Consider the continuous-time, free dynamic system

$$\frac{dx}{ds} = f(x,s) \qquad \text{.............................(B-19)}$$

where $f(0,s) = 0$ for all s.

Suppose there exists a scalar function V(x,s) with continuous first partial derivatives with respect to x and s such that V(0,s) = 0 and

(i) V(x,s) is positive definite; i.e., there exists a continuous, nondecreasing scalar function such that $\alpha(0) = 0$ and, for all s and all $x \neq 0$

$$0 < \alpha(\|x\|) \leq V(x,s) \qquad \text{(see Fig. B-1)}$$

(ii) There exists a continuous scalar function γ such that $\gamma(0) = 0$ and the derivative $\dot{V}$ of V along the motion starting at s,x satisfies, for all s and all x 0,

$$\begin{aligned}\dot{V}(x,s) &= \frac{dV(x,s)}{ds} \\ &= \lim_{h \to 0} \Big[V(x + hf(x,s), s + h) - V(x,s) \Big] / h \\ &= \frac{\partial V}{\partial s} + (\mathrm{grad} V)^T f(x,s) \\ &\leq -\gamma\big(\|x\|\big) < 0\end{aligned} \qquad \text{(B-20)}$$

(iii) There exists a continuous, nondecreasing scalar function such that $\beta(0) = 0$ and, for all s,

$$V(x,s) \leq \beta\big(\|x\|\big)$$

(iv) $\alpha\big(\|x\|\big) \longrightarrow \infty$ with $\|x\| \longrightarrow \infty$

Then the equilibrium state $x_e = 0$ is uniformly asymptotically stable in the large; V(x,s) is called a Lyapunov function of the system (B-21).

The following conditions are sufficient for the various weaker types of stability:

(a) Uniform asymptotic stability: (i-ii-iii)

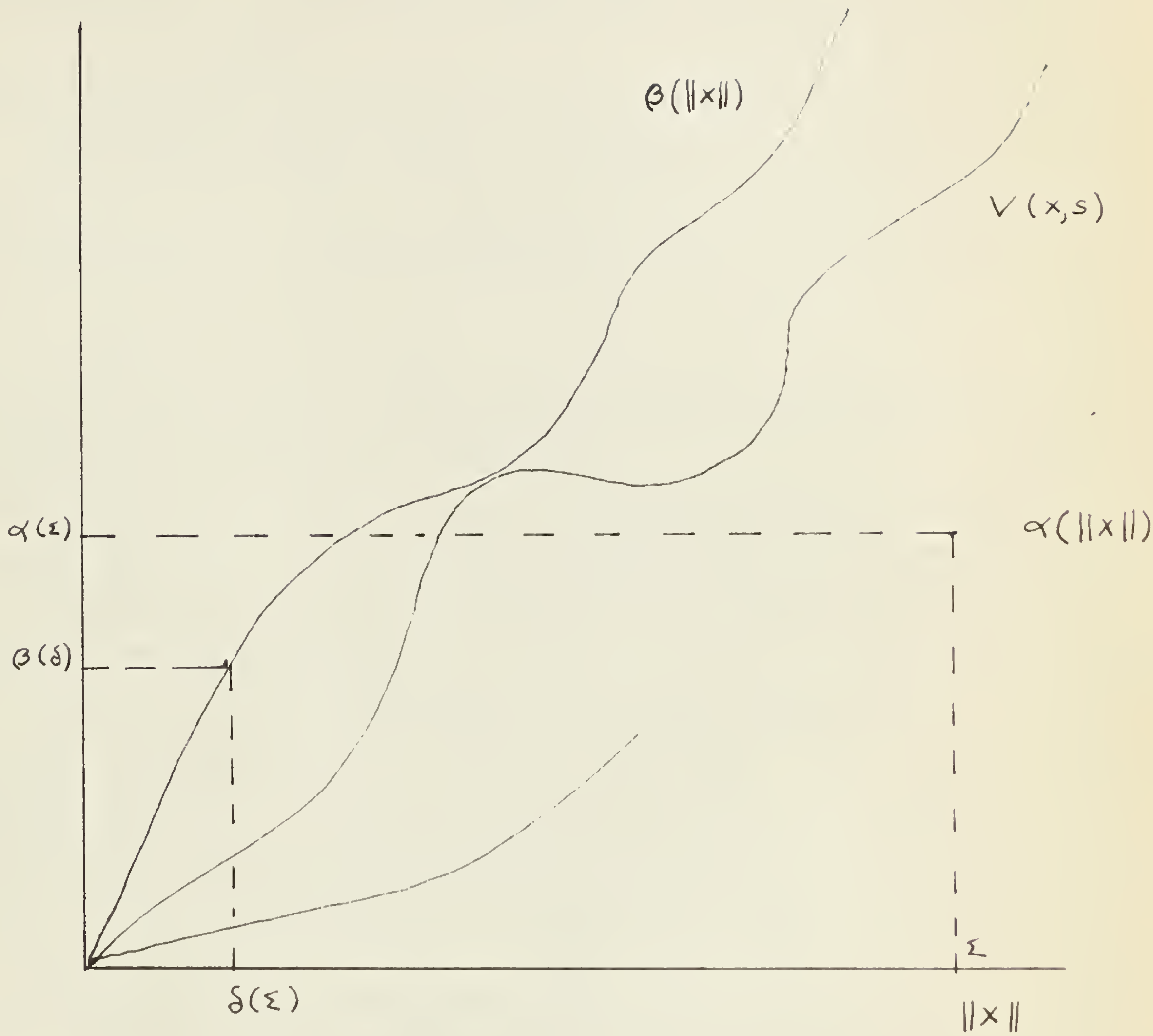
(b) Equiasymptotic stability in the large: (i-ii), (iv).

(c) Equiasymptotic stability: (i-ii).

(d) Uniform stability: (i),(iii), and $\dot{V}(x,s)$ 0 for all x,s

(e) Stability: (i) and $\dot{V}(x,s) \leq 0$ for all x,s.

For proofs see (11)



DEFINITIONS OF $V(x,s)$,

$\alpha(\|x\|)$, $\beta(\|x\|)$

APPENDIX C

Results from Wiener Equation

The Wiener Equation is given by

$$\text{cov}\quad x(t),z(\sigma) = \int_{t_o}^{t} A(t,\tau)\left[\text{cov}\left[z(\tau),z(\sigma)\right]\right.\, d\tau \text{..........(c-1)}$$

and is a necessary condition for an optimal system.

The optimal estimate $\hat{x}(t_1 \mid t)$ is given by

$$\hat{x}(t_1 \mid t) = \int_{t_o}^{t} A(t_1,\sigma)x(\sigma)d\sigma \text{ ....(c-2)}$$

where $t \leq \sigma \leq t_o$

with the property that the expected squared error in estimating any linear function of the message is minimized

$$\left[x^*_, x(t_1) - \hat{x}(t_1 \mid t)\right]^2 = \text{minimum for all } x^*$$

To develop the associated differential equations as given in section (3); proceed as follows.

let "$t_1 = t$" and differentiate (c-1) with respect to "t"

$$\text{L.H.S:}\ \frac{\partial}{\partial}\text{cov}\left[x(t),z(\sigma)\right] = \text{cov}\left[\frac{\partial}{\partial t}x(t),z(\sigma)\right] \text{ ....(c-3)}$$

and upon substitution of equation (B-6) into equation (c-3) we get

$$\text{L.H.S:}\quad F(t)\text{cov}\left[x(t),z(\sigma)\right] + G(t)\text{cov}\left[u(t),z(\sigma)\right] \text{ ......(c-4)}$$

$$\text{R.H.S:}\quad \frac{\partial}{\partial t}\int_{t_o}^{t} A(t,\tau)\ \text{cov}\left[z(\tau),z(\sigma)\right]\ d\tau$$

where from $z(t) = y(t) + v(t)$ we see that the

$$\text{cov}\left[\left[y(\tau) + v(\tau)\right]\left[y(\sigma) + v(\sigma)\right]\right] = \text{cov}\left[y(\tau),y(\sigma)\right] + \text{cov}\left[v(\tau),v(\sigma)\right] \text{ ....(c-5)}$$

because the system is uncorrelated i.e. cov $\left[y(\tau),v(\sigma)\right] = 0$

R.H.S: $\dfrac{\partial}{\partial t}\displaystyle\int_{t_o}^{t} A(t,\tau)\ \text{cov}\left[y(\tau),y(\sigma)\right]\ d\tau + \dfrac{\partial A(t,\sigma)}{\partial t}R(\sigma)$ ....(c-6)

$$\int_{t_o}^{t}\frac{\partial}{\partial t}A(t,\tau)\text{cov}\left[z(\tau),z(\sigma)\right]d\tau + A(t,t)\text{cov}\left[y(t),y(\sigma)\right] \quad \text{....(c-7)}$$

We see that the last term of equation (c-4)is zero because of the independence of $u(t),v(\sigma)$ and $x(\sigma)$ when $\sigma < t$

Further we note that by replacing $y(t)$ by $\left|z(t) - v(t)\right|$ the cov$\left[y(t),y(\sigma)\right]$ becomes

$$\text{cov}\left[(y(t),y(\sigma)\right] = \text{cov}\left[z(t),z(\sigma)\right] - \text{cov}\left[v(t),v(\sigma)\right] \quad \text{....(c-8)}$$

and using the fact that $y(t) = H(t)\ x(t)$ we further rearrange cov$\left[y(t),y(\sigma)\right]$ to be

$$\text{cov}\left[y(t),y(\sigma)\right] = \text{cov}\left[H(t)x(t),\ z(\sigma) - v(\sigma)\right]$$
$$= H(t)\text{cov}\left[x(t),z(\sigma)\right] - \text{cov}\left[y(t),v(\sigma)\right] \quad \text{...(c-9)}$$

We summarize important results at this point

From (c-4)we see that

$$\frac{\partial}{\partial t}\text{cov}\left[x(t),z(\sigma)\right] = F(t)\ \text{cov}\left[x(t),z(\sigma)\right]$$

From (c-1)the Wiener equation is

$$\text{cov}\left[x(t),z(\sigma)\right] = \int_{t_o}^{t} A(t,\tau)\text{cov}\left[z(\tau),z(\sigma)\right]d\tau$$

From (c-7)we see that $\dfrac{\partial}{\partial t}\text{cov}\left[x(t),z(\sigma)\right]$ is also

$$\int_{t_o}^{t}\frac{\partial}{\partial t}A(t,\tau)\text{cov}\left[z(\tau),z(\sigma)\right]d\tau + A(t,t)\text{cov}\left[y(t),y(\sigma)\right]$$

and from (c-9) we get

$$\text{cov}\left[y(t),y(\sigma)\right] = H(t)\text{cov}\left[x(t),z(\sigma)\right] \quad \text{where cov}\left[y(t),v(\sigma)\right] = 0$$

for $t > \sigma$

Now making use of the optimal estimate $\hat{x}(t \mid t) = \int_{t_o}^{t} \Big| A(t,\tau)$ $z(\tau)\Big| d\tau$ consider (I) equate (c-4)and (c-7)to get (c-10)

$$F(t)\mathrm{cov}\left[x(t),z(\sigma)\right] = \int_{t_o}^{t} \frac{\partial}{\partial t} A(t,\tau)\mathrm{cov}\left[z(\tau),z(\sigma)\right] d\tau +$$

$$A(t,t)H(t)\mathrm{cov}\left[x,z(\sigma)\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(c\text{-}10)$$

rearrange(c-10)and replace $\mathrm{cov}\left[x(t),z(\sigma)\right]$ by $\int_{t_o}^{t} \Big[A(t,\tau)$ $\mathrm{cov}\left[z(\tau),z(\sigma)\right]$ $d\tau$ from Wiener's Equation to get

$$\left[F(t) - A(t,t) \quad H(t)\right]\left[\int_{t_o}^{t} A(t,\tau)\mathrm{cov} \quad z(T)z(\quad) \quad dT\right.$$

$$d = \int_{t_o}^{t} \frac{\partial}{\partial t} A(t,\tau)\mathrm{cov}\left[z(\tau),z(\sigma)\right] d\tau \quad \ldots\ldots\ldots(c\text{-}11)$$

Taking everything under the integral and factoring out $\mathrm{cov}\left[z(\tau),\right.$ $\left.z(\sigma)\right]$ equation(c-11) becomes

$$\int_{t_o}^{t} \left[F(t)A(t,\tau) - A(t,t)H(t)A(t,\tau) - \frac{\partial}{\partial t}A(t,\tau)\right] \mathrm{cov}\Big[$$

$z(\tau),z(\sigma)\Big]$ $d\tau = 0$ because $\mathrm{cov}\left[z(\tau),z(\sigma)\right]$ is non zero we have that

$$\left[F(t)A(t,\tau) - \frac{\partial}{\partial}A(t,\tau) - A(t,t)H(t)A(t\tau)\right] = 0 \;\ldots\ldots\ldots(c\text{-}12)$$

for all values of $t_o \leq \tau \leq t$

If R(t) is positive definite in this interval then equation (c-13) is a necessary condition.

We now differentiate the optimal estimate to get

$$\frac{d}{dt}\left[\hat{x}(t \mid t)\right] = \frac{\partial}{\partial t}\int_{t_o}^{t} A(t,\tau)z(\tau)d\tau$$

$$= \int_{t_o}^{t} \frac{\partial}{\partial t} A(t,\tau)z(\tau)\, d\tau + A(t,t)z(t) \quad \ldots\ldots\ldots\ldots\ldots(c\text{-}15)$$

We now define $K(t) = A(t,t)$ ....(c-16)

and substitute the value for $\frac{\partial}{\partial t} A(t,\tau)$ from equation (c-13) into (c-15) resulting in

$$\frac{d\hat{x}(t|t)}{dt} = \int_{t_o}^{t} \left[F(t)A(t,\tau) - K(t)H(t)A(t,\tau)\ z(\tau) \right] d\tau \quad \text{....(c-17)}$$

$$+ K(t)z(t)$$

factor out $A(t,\tau)$ and notice that $\int_{t_o}^{t} A(t,\tau)z(\tau)\, d\tau$ occurs, so replace this with $\hat{x}(t\,|\,t)$, thus equation (c-17) becomes

$$\frac{d\hat{x}(t|t)}{dt} = F(t)\hat{x}(t|t) + K(t)\left[z(t) - H(t)\hat{x}(t\ t) \right] \quad \text{.........(c-18)}$$

this equation is denoted I in section (III)

We now let $\tilde{z}(t|t) = z(t) - H(t)\hat{x}(t|t)$ ....(c-19)

so I becomes

$$\frac{d}{dt}\hat{x}(t|t) = F(t)\hat{x}(t|t) + K(t)z(t|t)$$

To find the optimal error $\tilde{x}(t|t) = x(t) - \hat{x}(t|t)$ .....(c-20)

we differentiate $\tilde{x}(t\ t)$ with respect to "t" to get

$$\frac{d\tilde{x}(t|t)}{dt} = \frac{dx(t)}{dt} - \frac{d\hat{x}(t|t)}{dt} \quad \text{..............................(c-21)}$$

and from I and equation (B-6) we get

$$\frac{d\tilde{x}(t|t)}{dt} = F(t)\hat{x}(t|t) + K(t)\left[z(t) - H(t)\hat{x}(t|t) \right] - \left[F(t)x(t) - G(t)u(t) \right] \quad \text{..........................(c-22)}$$

$$= F(t)\tilde{x}(t|t) + G(t)u(t) - K(t)H(t)\tilde{x}(t\ t) - K(t)v(t)$$

which is denoted in section (III) as II

collecting terms we get

$$\frac{d\tilde{x}(t\ t)}{dt} = \left[F(t) - K(t)H(t) \right] x(t|t) - K(t)v(t) + G(t)u(t)$$

We now obtain an expression for K(t). To do this we observe from (c-23) that

$$\operatorname{cov}\left[x(t),y(\sigma)\right] = \int_{t_o}^{t} A(t,\tau)\operatorname{cov}\left[z(\tau),z(\sigma)\right] d\tau$$

and we also note that $\operatorname{cov}\left[\tilde{x}(t|t),z(\sigma)\right] = 0$ ....(c-24)

Equation(c-24)becomes from $z(\sigma) = y(\sigma) + v(\sigma)$

$$= \operatorname{cov}\left[\tilde{x}(t|t),y(\sigma)\right] + \operatorname{cov}\left[\tilde{x}(t|t),v(\sigma)\right] \quad \text{................(c-25)}$$

The second term is zero because of independence of $\tilde{x}(t|t)$ and $v(\sigma)$

We also know that $\operatorname{cov}\left[x(t),z(\sigma)\right] = \operatorname{cov}\left[x(t),y(\sigma)\right] + \operatorname{cov}\left[x(t),v(\sigma)\right]$ ....(c-26)

the second term of which is also zero because of independence of $x(t)$ and $v(\sigma)$.

Therefore the Wiener equation becomes

$$\operatorname{cov}\left[x(t),y(\sigma)\right] - \int_{t_o}^{t} A(t,\tau)\operatorname{cov}\left[y(\tau),y(\sigma)\right] d\tau = A(t,\sigma)R(\sigma) \quad \text{............(c-27)}$$

Since both sides are continuous functions of "σ" the equation must also hold when "$\sigma = t$" letting "$\sigma = t$" and equating equation (c-27) and equation (c-25) we get

$$\operatorname{cov}\left[\tilde{x}(t|t),y(t)\right] = A(t,t)R(t) \quad \text{.....................(c-29)}$$

because $y(t) = H(t)x(t) = x(t)H(t)^T$ Equation (c-29)becomes

$$K(t)R(t) = \operatorname{cov}\left[\tilde{x}(t|t),x(t)\right]H(t)^T \quad \text{........................(c-30)}$$

At this point we state and prove a corollary to Wiener's equation which will be used in the development of equations stated

in section (III)

Corollary ( )

The $\operatorname{cov}\left[\tilde{x}(t|t), \hat{x}(t|t)\right] = 0$

Proof:

From $\operatorname{cov}\left[\tilde{x}(t|t), z(\sigma)\right] = 0$ for all $t_o \leq \sigma \leq t$

and from $\hat{x}(t|t) = \int_{t_o}^{t} A(t,\tau) z(\tau)\, d\tau$ we see that

$$\left[\tilde{x}(t|t), \hat{x}(t|t)\right] = \int_{t_o}^{t} \tilde{x}(t|t) A(t,\tau) z(\tau)\, d\tau$$

and if A(t,) is symmetric we can write

$$\left[\tilde{x}(t|t), \hat{x}(t|t)\right] = \int_{t_o}^{t} \left[\tilde{x}(t|t), z(\tau) A(t,\tau)\right] d\tau$$

Taking the cov. of both sides we see that

$\operatorname{cov}\left[\tilde{x}(t|t), \hat{x}(t|t)\right] = 0$ Q.E.D.

From the above corollary we further see that

$$\operatorname{cov}\left[\tilde{x}(t|t), x(t)\right] = \operatorname{cov}\left[\tilde{x}(t|t), \tilde{x}(t|t)\right] \quad \text{......................(c-31)}$$

Upon substitution of equation(c-31) into equation(c-30) we get

$$K(t)R(t) = \operatorname{cov}\left[\tilde{x}(t|t), \tilde{x}(t|t)\right] H(t)^T \quad \text{..........................(c-32)}$$

Now define $P(t) = \operatorname{cov}\left[\tilde{x}(t|t), \tilde{x}(t|t)\right]$ ....(c-33)

therefore equation becomes

$$P(t)H(t)^T = K(t)R(t) \quad \text{......................(c-34)}$$

and if R(t) is positive definite

$K(t) = P(t)H(t)^T R(t)^{-1}$ which is denoted in section (III) by III

We now develop the Variance Equation

Summary

I $$\frac{d\hat{x}(t|t)}{dt} = F(t)\hat{x}(t|t) + K(t)\tilde{z}(t|t)$$

II $$\frac{d\tilde{x}(t|t)}{dt} = \left[F(t) - K(t)H(t)\right]\tilde{x}(t\ t) - K(t)v(t) + G(t)u(t)$$

III $$K(t) = P(t)H(t)^T R(t)^{-1}$$

We let $\Psi(t,\tau)$ be the common transition matrix of I and II

So the solution of this system will become

$$\tilde{x}(t|t) = \Psi(t,t_o)\tilde{x}(t_o\,|\,to) + \int_{t_o}^{t}\Psi(t,\tau)\left[G(\tau)u(\tau) - K(\tau)v(\tau)\right]d\tau \qquad \text{......................(c-35)}$$

We wish to establish an equation to determine P(t) therefore we must square equation(3-35) and take the cov. of both sides

$$\text{cov}\left[\tilde{x}(t|t) - \Psi(t,t_o)\tilde{x}(t_o\,|\,t_o)\right]\cdot\left[\tilde{x}(t|t) - \Psi(t,t_o)\tilde{x}(t_o\,|\,t_o)\right]^T$$

$$= \mathcal{E}\left[\int_{t_o}^{t}\Psi(t,\tau)\left[G(\tau)u(\tau) - K(\tau)v(\tau)\right]d\tau\right]\left[\int_{t_o}^{t}\Psi(t,\sigma)\left[G(\sigma)u(\sigma) - K(\sigma)v(\sigma)\right]d\sigma\right]^T \qquad \text{......................(c-36)}$$

$$= \int_{t_o}^{t}\Psi(t,\tau)\left[G(\tau)u(\tau) - K(\tau)v(\tau)\right]d\tau \int_{t_o}^{t}\left[u(\sigma)^T G(\sigma)^T - v(\sigma)^T K(\sigma)\right]\Psi(t,\sigma)^T\, d\sigma \qquad \text{................(c-37)}$$

because u(t) and v(t) are uncorrelated white noise the integral reduces to

$$P(t) - \Psi(t,t_o)P(t_o)\Psi(t,t_o)^T = \int_{t_o}^{t}\Psi(t,\tau)\left[G(\tau)Q(\tau)G(\tau)^T + K(\tau)R(\tau)K(\tau)^T\right]\Psi(t,\tau)^T\, d\tau \qquad \text{..............(c-38)}$$

let the bracket be C(t) and differentiate with respect to "t"

$$\int_{t_o}^{t} \frac{\partial}{\partial t} \Psi(t,\tau) \; C(\tau)\, \Psi(t,\tau)^T \, d\tau + \int_{t_o}^{t} \Psi(t,\tau) C(\tau)$$

$$\frac{\partial}{\partial t} \Psi(t,\tau)^T \, d\tau + \Psi(t,t) C(t)\, \Psi(t,t)^T \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (c\text{-}39)$$

where from

$$\frac{d\,\Psi(t,\tau)}{dt} = \left[F(t) - K(t)H(t) \right] \Psi(t,\tau) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (c\text{-}40)$$

and because $\Psi(t,t) = I$

$$\int_{t_o}^{t} \left[F(t) - K(t)H(t) \right] \Psi(t,\tau) C(\tau)\, \Psi(t,\tau)^T \, d\tau + C(t) +$$

$$\int_{t_o}^{t} \Psi(t,\tau) C(\tau)\, \Psi(t,\tau)^T \, d\tau \left[F(t)^T - H(t)^T K(t)^T \right] \ldots\ldots (C\text{-}41)$$

From equation (c-38)equation(c-41)becomes

$$= \left[F(t) - K(t)H(t) \right] \left[P(t) - \Psi(t,t_o) P(t_o)\, \Psi(t,t_o)^T \right] + C(t)$$
$$+ \left[P(t) - \Psi(t,t_o) P(t_o)\, \Psi(t,t_o)^T \right] \left[F(t)^T - H(t)^T K(t) \right] \ldots (C\text{-}42)$$

We now differentiate the L.H.S.

$$\frac{dP(t)}{dt} - \frac{\partial}{\partial t} \Psi(t,t_o) \; P(t_o)\, \Psi(t,t_o)^T - \Psi(t,t_o) P(t_o)$$

$$\frac{\partial}{\partial t} \Psi(t,t_o)^T = \frac{dP(t)}{dt} - \left[F(t) - K(t)H(t) \right] \left[\Psi(t,t_o) P(t_o) \right.$$
$$\left. \Psi(t,t_o)^T \right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (c\text{-}43)$$
$$- \Psi(t,t_o) P(t_o)\, \Psi(t,t_o)^T \left[F(t)^T - H(t)^T K(t)^T \right]$$

Equate equations (c-43)and equations(c-42)and cancel terms results in

$$\frac{dP(t)}{dt} = F(t)P(t) + P(t)F(t)^T - K(t)H(t) - P(t)H(t)^T K(t)^T +$$

$$K(t)R(t)K(t)^T + G(t)Q(t)G(t)^T \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c\text{-}44)$$

This equation is denoted as IV in section (III)

We now consider the problem of prediction and find a form for the optimal estimate problem when "$t_1 > t$"

The solution of the equation $\frac{dx(t)}{dt} = F(t)x(t) + G(t)u(t)$

is given by

$$\phi(t_1;t,x(t)) = \Phi(t_1,t)x(t) + \int_t^{t_1} \Phi(t_1, \xi)G(\xi)u(\xi)d\xi$$

Since $u(\xi)$ for $t < \xi \leq t_1$ is independent of $x(t)$ in the interval $t_0 \leq \xi \leq t$ it follows from Wiener's equation that the optimal estimator for the right-hand side above is zero. Hence

$$\hat{x}(t_1 \mid t) = \Phi(t_1,t)\hat{x}(t_1 \mid t) \quad (t_1 > t) \quad \text{..................(c-46)}$$

This however is not true for $t_1 < t$ because of the lack of independence between $x(\xi)$ and $u(\xi)$.

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